

FRM Part I — Complete Formula Sheet

All four books · Foundations · Quantitative Analysis · Financial Markets & Products · Valuation & Risk Models

Prepared by **RBei Classes**, Noida · Based on the GARP FRM Part I curriculum

How to use this sheet. Formulas are grouped by book and topic. Conventions used throughout: r = risk-free rate (continuously compounded unless stated), T = time in years, S_0 = spot price, X or K = strike / delivery price, σ = volatility (standard deviation), ρ = correlation, $N(\cdot)$ = standard normal CDF, $N^{-1}(\cdot)$ = its inverse. **Orange boxes** flag common exam traps.

Book 1 — Foundations of Risk Management

Exam weight: 20%

Expected Loss, Economic Capital & RAROC

$$EL = PD \times LGD \times EAD, \quad LGD = 1 - RR$$

PD = probability of default, LGD = loss given default, EAD = exposure at default, RR = recovery rate.

$$\text{Unexpected loss (UL)} = \text{Actual loss} - EL$$

$$\text{Economic capital} = \text{VaR}_{X\%, 1\text{yr}} - EL$$

$$\text{RAROC} = \frac{\text{After-tax risk-adjusted expected return}}{\text{Economic capital}}$$

Accept an activity only if RAROC > cost of equity (hurdle rate).

Basel ratios (qualitative, but know the numbers)

- Basel I minimum capital = 8% of risk-weighted assets.
- Leverage ratio = $\frac{\text{Tier 1 capital}}{\text{Total exposure}} \geq 3\%$
- LCR = $\frac{\text{HQLA}}{\text{Net cash outflows over 30 days}} \geq 100\%$
- NSFR = $\frac{\text{Available stable funding}}{\text{Required stable funding}} \geq 100\%$ (1-year horizon)
- Securitisation originator must retain $\geq 5\%$ of credit risk (SEC Sec. 15G).

Two-asset portfolio

$$E(R_P) = w_A E(R_A) + w_B E(R_B)$$

$$\sigma_P^2 = w_A^2 \sigma_A^2 + w_B^2 \sigma_B^2 + 2w_A w_B \rho_{AB} \sigma_A \sigma_B$$

n assets: $\sigma_P^2 = \sum_i \sum_j w_i w_j \rho_{ij} \sigma_i \sigma_j$. With $\rho = -1$ a zero-variance portfolio is possible.

CML, Beta, CAPM / SML

Capital Market Line (efficient portfolios only):

$$E(R_P) = R_F + \left[\frac{E(R_M) - R_F}{\sigma_M} \right] \sigma_P$$

Slope of CML = Sharpe ratio of the market.

Beta:

$$\beta_i = \frac{\text{Cov}(R_i, R_M)}{\sigma_M^2} = \rho_{i,M} \frac{\sigma_i}{\sigma_M}, \quad \beta_P = \sum w_i \beta_i, \quad \beta_M = 1$$

CAPM (Security Market Line):

$$E(R_i) = R_F + \beta_i [E(R_M) - R_F]$$

Slope of SML = market risk premium $E(R_M) - R_F$. Forecast return > CAPM return $\Rightarrow$ undervalued (plots above SML).

Performance measures

$$\text{Sharpe} = \frac{E(R_P) - R_F}{\sigma_P} \quad (\text{total risk})$$

$$\text{Treynor} = \frac{E(R_P) - R_F}{\beta_P} \quad (\text{systematic risk})$$

$$\text{Jensen's } \alpha = E(R_P) - \{R_F + \beta_P [E(R_M) - R_F]\}$$

$$\text{Tracking error} = TE = \sigma(R_P - R_B)$$

$$\text{Information ratio} = IR = \frac{E(R_P) - E(R_B)}{TE}$$

$$\text{Sortino} = \frac{E(R_P) - R_{MIN}}{\text{Downside deviation}}$$

Downside deviation = standard deviation of only those returns falling below R_{MIN} (minimum acceptable return).

Trap: Treynor > market but Sharpe < market $\Rightarrow$ portfolio is **not well-diversified**.

APT, multifactor & Fama-French

Multifactor return model (F = factor *surprise*):

$$R_i = E(R_i) + \beta_{i1} F_1 + \beta_{i2} F_2 + \dots + \beta_{ik} F_k + e_i$$

Surprise F = actual value of factor – expected value. $E(e_i) = 0$.

APT expected return:

$$E(R_i) = R_F + \beta_{i1} RP_1 + \beta_{i2} RP_2 + \dots + \beta_{ik} RP_k$$

Fama-French three-factor model:

$$E(R_i) = R_F + \beta_{i,M} [E(R_M) - R_F] + \beta_{i,SMB} E(SMB) + \beta_{i,HML} E(HML)$$

SMB = small minus big (size); HML = high minus low book-to-market (value). Carhart adds momentum; FF-5 adds RMW and CMA.

Book 2 — Quantitative Analysis

Exam weight: 20%

Probability rules

$$P(A \cup B) = P(A) + P(B) - P(AB)$$

$$P(AB) = P(A | B)P(B), \quad P(A | B) = \frac{P(AB)}{P(B)}$$

Independent: $P(AB) = P(A)P(B)$ and $P(A | B) = P(A)$. **Mutually exclusive:** $P(AB) = 0$.

Total probability (B_i mutually exclusive & exhaustive):

$$P(A) = \sum_i P(A | B_i)P(B_i)$$

Bayes' rule:

$$P(A | B) = \frac{P(B | A)P(A)}{P(B)}$$

Moments of a random variable

$$E(X) = \mu = \sum x_i P(x_i)$$

$$\sigma^2 = E[(X - \mu)^2] = E(X^2) - [E(X)]^2$$

$$\text{Skewness} = \frac{E[(X - \mu)^3]}{\sigma^3}, \quad \text{Kurtosis} = \frac{E[(X - \mu)^4]}{\sigma^4}$$

Normal: skew = 0, kurtosis = 3. Excess kurtosis = kurtosis - 3. Positive skew: mode < median < mean.

Linear transformation $Y = a + bX$:

$$E(Y) = a + bE(X), \quad \sigma_Y^2 = b^2\sigma_X^2, \quad \sigma_Y = |b|\sigma_X$$

Skew unchanged if $b > 0$, sign flips if $b < 0$. Kurtosis unchanged.

Common distributions

Dist.	PMF / PDF	Mean; Var
Uniform	$P(x_1 \leq X \leq x_2) = \frac{x_2 - x_1}{b - a}$	$\frac{a+b}{2}; \frac{(b-a)^2}{12}$
Bernoulli	$p^x(1-p)^{1-x}$	$p; p(1-p)$
Binomial	$\binom{n}{x} p^x(1-p)^{n-x}$	$np; np(1-p)$
Poisson	$\frac{\lambda^x e^{-\lambda}}{x!}$	$\lambda; \lambda$
Exponential	$\frac{1}{\beta} e^{-x/\beta}, \lambda = 1/\beta$	$\beta; \beta^2$
Normal	$z = \frac{x - \mu}{\sigma}$	$\mu; \sigma^2$

$\binom{n}{x} = \frac{n!}{x!(n-x)!}$. Lognormal: $\ln X$ is normal; right-skewed, bounded below by 0. Poisson count of defaults up to t has rate t/β .

t : $df = n - 1$, fatter tails than normal. χ^2 and F : right-skewed, bounded at 0.

Normal-distribution key values

Confidence	90%	95%	99%
One-tailed z	1.28	1.645	2.33
Two-tailed z	1.65	1.96	2.58

$\pm 1\sigma \approx 68\%$, $\pm 2\sigma \approx 95\%$, $\pm 3\sigma \approx 99.7\%$. $N(-z) = 1 - N(z)$.

Multivariate: covariance & correlation

$$\text{Cov}(X, Y) = E\{[X - E(X)][Y - E(Y)]\} = E(XY) - E(X)E(Y)$$

$$\rho_{XY} = \frac{\text{Cov}(X, Y)}{\sigma_X \sigma_Y}, \quad -1 \leq \rho \leq 1$$

$$\text{Cov}(a + bX, c + dY) = bd \text{Cov}(X, Y)$$

$$\text{Var}(aX + bY) = a^2\sigma_X^2 + b^2\sigma_Y^2 + 2ab \text{Cov}(X, Y)$$

Minimum-variance weight (asset 1):

$$w_1^* = \frac{\sigma_2^2 - \sigma_{12}}{\sigma_1^2 + \sigma_2^2 - 2\sigma_{12}}$$

Conditional: $P(X_1 = x_1 | X_2 = x_2) = \frac{f(x_1, x_2)}{f_{X_2}(x_2)}$; $E(X_1 | X_2) = \sum x_1 P(x_1 | x_2)$

i.i.d. sums: $E(\sum X_i) = n\mu$, $\text{Var}(\sum X_i) = n\sigma^2$, $\text{Var}(\bar{X}) = \sigma^2/n$

$\rho = 0$ does **not** imply independence (only no *linear* relation).

Sample moments

$$\hat{\mu} = \bar{X} = \frac{1}{n} \sum X_i$$

$$s^2 = \frac{1}{n-1} \sum (X_i - \bar{X})^2 \text{ (unbiased)} \quad \hat{\sigma}^2 = \frac{1}{n} \sum (X_i - \bar{X})^2 \text{ (biased)}$$

$$E(\hat{\sigma}^2) = \frac{n-1}{n} \sigma^2, \quad \text{Bias}(\hat{\theta}) = E(\hat{\theta}) - \theta$$

$$\hat{\sigma}_{XY} = \frac{1}{n-1} \sum (X_i - \bar{X})(Y_i - \bar{Y})$$

Median: n odd: $x_{(n+1)/2}$; n even: $\frac{1}{2}(x_{n/2} + x_{n/2+1})$ (sorted data)

CLT: $\bar{X} \sim N(\mu, \sigma^2/n)$ for large n (needs finite variance). Sample mean is **BLUE**.

Hypothesis testing

$$\text{Test stat} = \frac{\text{Sample statistic} - \text{Hypothesised value}}{\text{Standard error}}$$

$$SE(\bar{X}) = \frac{\sigma}{\sqrt{n}} \text{ or } \frac{s}{\sqrt{n}}, \quad t = \frac{\bar{X} - \mu_0}{s/\sqrt{n}} \quad (df = n - 1)$$

$$\text{CI: } \bar{X} \pm t_{\alpha/2} \frac{s}{\sqrt{n}} \quad (\text{two-sided})$$

Difference of two means (paired, same n):

$$T = \frac{\hat{\mu}_X - \hat{\mu}_Y}{\sqrt{(\hat{\sigma}_X^2 + \hat{\sigma}_Y^2 - 2\hat{\sigma}_{XY})/n}}$$

Variance tests: $\chi^2 = \frac{(n-1)s^2}{\sigma_0^2}$ ($df = n - 1$); $F = \frac{s_1^2}{s_2^2}$

Errors: $P(\text{Type I}) = \alpha$ (size); Power = $1 - P(\text{Type II})$.

Reject H_0 if $|\text{test stat}| > \text{critical value}$, or $p\text{-value} < \alpha$. Multiple testing inflates the Type I error.

Linear regression (single)

$$Y_i = \alpha + \beta X_i + \varepsilon_i$$

$$\hat{\beta} = \frac{\text{Cov}(X, Y)}{\text{Var}(X)} = \frac{\sum (X_i - \bar{X})(Y_i - \bar{Y})}{\sum (X_i - \bar{X})^2}, \quad \hat{\alpha} = \bar{Y} - \hat{\beta} \bar{X}$$

$$t = \frac{\hat{\beta} - \beta_0}{SE(\hat{\beta})}, \quad df = n - 2; \quad \text{CI} = \hat{\beta} \pm t_c SE(\hat{\beta})$$

$$R^2 = \rho_{XY}^2 \quad (\text{single regressor only})$$

Multiple regression & fit

$$Y_i = \alpha + \beta_1 X_{1i} + \dots + \beta_k X_{ki} + \varepsilon_i$$

$$\underbrace{\sum (Y_i - \bar{Y})^2}_{TSS} = \underbrace{\sum (\hat{Y}_i - \bar{Y})^2}_{ESS} + \underbrace{\sum (Y_i - \hat{Y}_i)^2}_{RSS}$$

$$R^2 = \frac{ESS}{TSS} = 1 - \frac{RSS}{TSS}$$

$$\bar{R}^2 = 1 - (1 - R^2) \frac{n-1}{n-k-1} \quad (\bar{R}^2 \leq R^2, \text{ can be } < 0)$$

$$SER = \sqrt{\frac{RSS}{n-k-1}}$$

t -test on each coefficient uses $df = n - k - 1$.

F-test, all slopes = 0:

$$F = \frac{ESS/k}{RSS/(n-k-1)}$$

F-test, full vs. restricted model (q restrictions):

$$F = \frac{(RSS_R - RSS_U)/q}{RSS_U/(n - k_U - 1)}$$

F-tests are always one-tailed.

Regression diagnostics

White test (heteroskedasticity): regress $\hat{\varepsilon}^2$ on regressors (levels, squares, cross-products):

$$\chi^2 = nR^2, \quad df = \frac{k(k+3)}{2}$$

Variance inflation factor:

$$VIF_j = \frac{1}{1 - R_j^2} \quad (VIF > 10 \Rightarrow \text{multicollinearity})$$

Cook's distance:

$$D_j = \frac{\sum_i (\hat{Y}_i^{(-j)} - \hat{Y}_i)^2}{k s^2} \quad (D_j > 1 \Rightarrow \text{outlier})$$

Conditional heteroskedasticity: coefficients unbiased, SEs unreliable $\Rightarrow$ use White SEs.
Omitted variable bias needs: omitted var. correlated with an included X and a determinant of Y .

Stationary time series

$$\gamma(\tau) = \text{Cov}(Y_t, Y_{t-\tau}), \quad \rho(\tau) = \frac{\gamma(\tau)}{\gamma(0)}$$

AR(1): $Y_t = \delta + \phi Y_{t-1} + \varepsilon_t$ ($|\phi| < 1$)

$$\mu = \frac{\delta}{1 - \phi}, \quad \gamma(0) = \frac{\sigma^2}{1 - \phi^2}, \quad \rho(\tau) = \phi^{|\tau|}$$

Mean-reverting level = $\delta / (1 - \phi)$. AR(p) mean = $\delta / (1 - \phi_1 - \dots - \phi_p)$. PACF cuts off after lag p .

MA(1): $Y_t = \mu + \theta \varepsilon_{t-1} + \varepsilon_t$

$$E(Y_t) = \mu, \quad \gamma(0) = (1 + \theta^2)\sigma^2, \quad \rho(1) = \frac{\theta}{1 + \theta^2}, \quad \rho(\tau) = 0 \quad (\tau \geq 2)$$

Portmanteau tests ($\sim \chi_h^2$):

$$Q_{BP} = T \sum_{\tau=1}^h \hat{\rho}^2(\tau), \quad Q_{LB} = T(T+2) \sum_{\tau=1}^h \frac{\hat{\rho}^2(\tau)}{T - \tau}$$

Non-stationary time series

Linear trend: $Y_t = \delta_0 + \delta_1 t + \varepsilon_t$; Log-linear: $\ln Y_t = \delta_0 + \delta_1 t + \varepsilon_t$

Seasonal dummies: $Y_t = \delta_0 + \sum_{j=1}^{s-1} \gamma_j D_{jt} + \varepsilon_t$

Random walk (unit root): $Y_t = Y_{t-1} + \varepsilon_t$

95% interval forecast: $\hat{Y}_{T+h} \pm 1.96 \hat{\sigma}$

Use $s - 1$ dummies with an intercept (avoid dummy trap). Unit root tested with ADF test; fix by differencing.

Returns, volatility & correlation

$$R_t = \frac{P_t - P_{t-1}}{P_{t-1}}, \quad r_t = \ln \frac{P_t}{P_{t-1}} = \ln(1 + R_t)$$

$$1 + R_T = \prod_{t=1}^T (1 + R_t), \quad r_T = \sum_{t=1}^T r_t$$

$$\sigma_{T\text{-period}} = \sigma_{1\text{-period}} \sqrt{T} \quad (\text{annual} = \text{daily} \times \sqrt{252})$$

Jarque-Bera ($\sim \chi_2^2$; 5% critical = 5.99):

$$JB = (T - 1) \left[\frac{\hat{S}^2}{6} + \frac{(\hat{K} - 3)^2}{24} \right]$$

Power law: $P(X > x) = kx^{-\alpha}$ (smaller α = fatter tail)

$$\text{Spearman: } \rho_s = 1 - \frac{6 \sum d_i^2}{n(n^2 - 1)} \quad \text{Kendall: } \tau = \frac{n_c - n_d}{n(n - 1)/2}$$

One-factor model: $U_i = a_i F + \sqrt{1 - a_i^2} Z_i \Rightarrow \rho(U_i, U_j) = a_i a_j$

Simulation

$$SE(\hat{\mu}) = \frac{\hat{\sigma}}{\sqrt{n}} \quad (\text{halve SE} \Rightarrow 4 \times \text{replications})$$

Antithetic variates: pair each draw with its negative (works when $\rho < 0$ between pairs).
Control variates: use a correlated variable with known mean. Bootstrapping resamples actual data with replacement; fails with structural breaks or small samples.

Machine learning

$$\text{Standardisation: } \frac{x - \mu}{\sigma} \quad \text{Normalisation: } \frac{x - x_{\min}}{x_{\max} - x_{\min}}$$

$$\text{Logistic regression: } P(Y=1) = \frac{1}{1 + e^{-(\alpha + \beta_1 X_1 + \dots + \beta_k X_k)}}$$

Ridge: $\min RSS + \lambda \sum \beta_j^2$ **LASSO:** $\min RSS + \lambda \sum |\beta_j|$

Decision trees: Gini = $1 - \sum p_i^2$; Entropy = $-\sum p_i \log_2 p_i$; Info gain = parent entropy – weighted child entropy.

Q-learning update: $Q(s, a) \leftarrow Q(s, a) + \alpha [R + \gamma \max_{a'} Q(s', a') - Q(s, a)]$

Confusion matrix:

$$\text{Accuracy} = \frac{TP + TN}{\text{Total}}, \quad \text{Precision} = \frac{TP}{TP + FP}, \quad \text{Recall} = \frac{TP}{TP + FN}$$

Error rate = $1 - \text{accuracy}$. Overfitting: low bias, high variance. K -means minimises within-cluster distance.

Book 3 — Financial Markets & Products

Exam weight: 30%

Insurance companies

P&C ratios (all as % of premiums):

- Loss ratio = claims paid / premiums
- Expense ratio = expenses / premiums
- Combined ratio = loss ratio + expense ratio
- Combined after dividends = combined + dividends
- Operating ratio = combined after dividends – investment income

Mortality: $P(\text{die in yr } 2) = (1 - q_1) \times q_2$ (q = 1-yr death probability)**Break-even premium:** $PV(\text{expected premiums}) = PV(\text{expected payouts})$; 2nd-year premium is received only with survival probability $(1 - q_1)$.

Fund management

$$NAV = \frac{\text{Market value of assets} - \text{Liabilities}}{\text{Shares outstanding}}$$

Hedge fund “2 and 20” (mgmt fee on start-of-year assets, incentive after mgmt fee):

$$\text{Incentive} = 20\% \times \max[0, \text{Gross return} - 2\% - \text{Hurdle}]$$

$$\text{Investor return} = \text{Gross return} - \text{Mgmt fee} - \text{Incentive fee}$$

High-water mark: past losses must be recovered before incentive fees. Clawback: earlier fees returned after later losses.

Option & forward payoffs

Position	Payoff at T	Profit
Long call	$\max(0, S_T - X)$	$\max(0, S_T - X) - c_0$
Long put	$\max(0, X - S_T)$	$\max(0, X - S_T) - p_0$
Long forward	$S_T - K$	—
Short forward	$K - S_T$	—

Writers: negative of buyer. Breakeven: call $X + c_0$, put $X - p_0$. Moneyness (call): ITM $S > X$, ATM $S = X$, OTM $S < X$.

Margin

Futures: daily MTM gain/loss = $\Delta F \times \text{contract size} \times \#\text{contracts}$. If balance < maintenance margin $\Rightarrow$ top up to **initial** margin (variation margin).**Short sale:** initial = 150% of value; maintenance = 125%; call restores to maintenance.**Buying on margin:** Margin % = $\frac{\text{Equity}}{\text{Market value}}$; call if < maintenance.**Writing an option** (per option), margin = greater of:

- (1) $100\% \times \text{option price} + 20\% \times S - \text{OTM amount}$
- (2) $100\% \times \text{option price} + 10\% \times S$ (call) or $10\% \times X$ (put)

Hedging with futures

Basis = Spot price of asset hedged – Futures price used

$$h^* = \rho_{S,F} \frac{\sigma_S}{\sigma_F} = \frac{\text{Cov}(\Delta S, \Delta F)}{\sigma_F^2}, \quad \text{Hedge effectiveness} = \rho^2$$

$$N^* = h^* \frac{Q_A}{Q_F} \quad \text{Tailing: } N^* = h^* \frac{V_A}{V_F}$$

 Q_A = units hedged, Q_F = units per contract; V_A = value of position, V_F = futures price $\times$ contract size.**Stock-index hedge / change beta:**

$$N = \beta \frac{P}{A}, \quad N = (\beta^* - \beta) \frac{P}{A}$$

 P = portfolio value, A = futures price $\times$ multiplier. $N < 0 \Rightarrow$ sell futures.**Effective price:** short hedge receives $S_2 + (F_1 - F_2) = F_1 + b_2$.

Foreign exchange

Quote XXXYYY = units of YYY (quote ccy) per 1 XXX (base ccy). Bid < ask.

Forward outright = spot + points $\times 0.0001$ **Covered interest rate parity:**

$$F = S \times \frac{(1 + r_{\text{quote}})^T}{(1 + r_{\text{base}})^T} \quad \text{or} \quad F = S e^{(r_{\text{quote}} - r_{\text{base}})T}$$

Relative PPP: $\% \Delta S_{(\text{quote/base})} \approx \pi_{\text{quote}} - \pi_{\text{base}}$ **Fisher:** $(1 + r) = (1 + r_{\text{real}})(1 + E(\pi))$, $r \approx r_{\text{real}} + E(\pi)$ **% appreciation of base ccy** = $S_1/S_0 - 1$ (quote ccy change must be computed on the inverted quote).Higher interest-rate currency trades at a forward **discount**. Uncovered IRP: expected spot change offsets the rate differential.

Pricing financial forwards & futures

Underlying	Forward price F_0	Value of long (K)
No income	$S_0 e^{rT}$	$S_0 - K e^{-rT}$
Known income I (PV)	$(S_0 - I) e^{rT}$	$S_0 - I - K e^{-rT}$
Known yield q	$S_0 e^{(r-q)T}$	$S_0 e^{-qT} - K e^{-rT}$

General: value of long forward = $(F_0 - K) e^{-rT}$. Stock index: q = dividend yield. Currency: $q = r_f$.**Short sale P&L** = (sale price – buy-back price) – dividends paid to lender.If actual $F > S_0 e^{rT}$: cash-and-carry (buy spot, borrow, short forward). If $F < S_0 e^{rT}$: reverse cash-and-carry.

Commodity forwards

$$F_0 = S_0 e^{(r - \delta_l)T} \quad (\delta_l = \text{lease rate})$$

$$F_0 = (S_0 + U) e^{rT} \quad (U = \text{PV of storage}); \quad F_0 = S_0 e^{(r+u)T} \quad (u = \text{storage rate})$$

$$F_0 = S_0 e^{(r+u-y)T} \quad (y = \text{convenience yield})$$

No-arbitrage: $F_0 \leq S_0 e^{(r+u)T}$. Contango: $F > S$ (upward curve). Backwardation: $F < S$. Normal backwardation: $F < E(S_T)$; normal contango: $F > E(S_T)$.

Option bounds & put–call parity

	Lower bound	Upper
Euro call	$\max(0, S_0 - X e^{-rT})$	S_0
Euro put	$\max(0, X e^{-rT} - S_0)$	$X e^{-rT}$
Amer. call	$\max(0, S_0 - X e^{-rT})$	S_0
Amer. put	$\max(0, X - S_0)$	X

With dividends replace S_0 by $S_0 - D$ (PV of dividends).**Put–call parity (European):**

$$c + X e^{-rT} = p + S_0 \quad (\text{with dividends: } c + D + X e^{-rT} = p + S_0)$$

$$c - p = (F_0 - X) e^{-rT} \quad (\text{forward form})$$

American bounds:

$$S_0 - X \leq C - P \leq S_0 - X e^{-rT}; \quad S_0 - D - X \leq C - P \leq S_0 - X e^{-rT}$$

Never optimal to exercise an American call on a non-dividend stock early.

Six factors affecting option prices

↑ Factor	Eu. c	Eu. p	Am. C	Am. P
Stock price S	+	–	+	–
Strike X	–	+	–	+
Time T	?	?	+	+
Volatility σ	+	+	+	+
Risk-free r	+	–	+	–
Dividends D	–	+	–	+

Trading strategies

Strategy	Construction
Covered call	Long stock + short call
Protective put	Long stock + long put
Bull spread	Long call X_1 , short call X_2 ($X_1 < X_2$)
Bear spread	Long put X_2 , short put X_1 ($X_1 < X_2$)
Butterfly	Long X_1 & X_3 calls, short 2 X_2 calls
Box spread	Bull call spread + bear put spread; value = $(X_2 - X_1)e^{-rT}$
Straddle	Long call + long put, same X
Strangle	Long put X_1 + long call X_2
Strip / Strap	1C+2P / 2C+1P, same X

Bull call spread: max profit = $X_2 - X_1$ - net premium; max loss = net premium. Calendar spread: same X , different maturities. PPN: zero-coupon bond + call option.

Interest rates & compounding

$$FV = A \left(1 + \frac{R_m}{m} \right)^{mm}, \quad FV = Ae^{R_c n}$$
$$R_c = m \ln \left(1 + \frac{R_m}{m} \right), \quad R_m = m \left(e^{R_c/m} - 1 \right)$$

Forward rate (continuous):
$$R_F = \frac{R_2 T_2 - R_1 T_1}{T_2 - T_1} = R_2 + (R_2 - R_1) \frac{T_1}{T_2 - T_1}$$

Forward rate (annual compounding): $(1 + R_2)^{T_2} = (1 + R_1)^{T_1} (1 + F)^{T_2 - T_1}$

FRA payoff (to receiver of fixed R_K), paid at T_2 :
$$L(R_K - R_M)(T_2 - T_1); \quad \text{Value} = L(R_K - R_F)(T_2 - T_1)e^{-R_2 T_2}$$

Bond price: $B = \sum_t c_t e^{-y t}$ (or discount each cash flow at its spot rate).
Bootstrapping: solve spot rates sequentially from shortest bond. Theories: expectations, market segmentation, liquidity preference.

Duration & convexity (FMP)

$$D_{Mac} = \frac{\sum t_i PV(c_i)}{B}, \quad D_{Mod} = \frac{D_{Mac}}{1 + y/m}$$
$$\Delta B \approx -D_{Mod} B \Delta y + \frac{1}{2} C B (\Delta y)^2$$
$$C = \frac{1}{B} \sum t_i^2 c_i e^{-y t_i} \quad (\text{continuous})$$

Dollar duration = $D_{Mod} \times B$, $DV01 = \text{Dollar duration} \times 0.0001$

With continuous compounding $D_{Mac} = D_{Mod}$. Convexity is always positive for option-free bonds.

Corporate bonds

$$\text{Expected loss rate} = \text{Default rate} \times (1 - \text{Recovery rate})$$
$$\text{Issuer default rate} = \frac{\# \text{ issuers defaulted}}{\# \text{ issuers at start}}, \quad \text{Dollar default rate} = \frac{\text{Par of defaulted}}{\text{Total par}}$$
$$\text{Expected return} \approx \text{yield} - \text{expected loss rate}.$$

Mortgages & MBS

$$\text{Monthly payment} = P_0 \frac{r(1+r)^n}{(1+r)^n - 1} \quad (r = \text{annual rate}/12, n = \text{months})$$

Interest = $r \times$ opening balance; principal = payment - interest.

$$SMM = 1 - (1 - CPR)^{1/12}, \quad CPR = 1 - (1 - SMM)^{12}$$
$$\text{Prepayment}_t = SMM \times (\text{Opening balance} - \text{Scheduled principal})$$

WAC and WAM are balance-weighted averages of coupons and maturities. PSA 100: CPR rises 0.2%/month to 6% at month 30. OAS = spread over the risk-free curve after removing option cost.

Day counts & Treasury futures

$$\text{Interest} = \text{Coupon} \times \frac{\text{Days in accrual period}}{\text{Days in reference period}}$$

Actual/actual: US Treasuries. 30/360: US corporate & municipal. Actual/360: money market (T-bills).

T-bill: $P = 100 - \frac{n}{360} Y$ (Y = discount-rate quote, n = days)

Dirty (cash) price = clean (quoted) price + accrued interest

Cash received by short (delivery):
$$(QFP \times CF) + AI$$

Cost of delivering: Quoted bond price - $(QFP \times CF) \Rightarrow$ CTD minimises this.

Theoretical futures: $F_{\text{cash}} = (S_{\text{cash}} - I)e^{rT}$; then quoted = $(F_{\text{cash}} - AI_T)/CF$.

Duration-based hedge ratio:
$$N^* = \frac{P \times D_P}{F_C \times D_F}$$

P = portfolio value, F_C = futures contract price, D = durations. Conversion factor: price of \$1 of bond at 6% yield (semiannual).

Swaps

Bond approach (pay-fixed side):
$$V_{\text{swap}} = B_{\text{float}} - B_{\text{fix}}, \quad B_{\text{float}} = (L + k^*)e^{-r_1 t_1}$$

L = notional, k^* = next floating payment (already set), t_1 = time to next reset. Receive-fixed = $B_{\text{fix}} - B_{\text{float}}$.

FRA approach: value = $\sum$ PV of [forward-rate floating payment - fixed payment].

Currency swap (receive domestic, pay foreign):
$$V = B_D - S_0 B_F$$

S_0 = domestic per unit foreign. Forward approach: $\sum$ PV of each exchange using $F_t = S_0 e^{(r-r_f)t}$.

Book 4 — Valuation & Risk Models

Exam weight: 30%

VaR & Expected Shortfall (normal)

$$VaR_X = (z_X \sigma - \mu) \times \text{Value}$$

$$ES_X = \left[\sigma \frac{e^{-z_X^2/2}}{\sqrt{2\pi}(1-X)} - \mu \right] \times \text{Value}$$

X = confidence level; z_X = one-tailed z (1.645 at 95%, 2.326 at 99%). With $\mu = 0$: $ES_{95} = 2.063\sigma$, $ES_{99} = 2.665\sigma$.

$$VaR_T = VaR_1 \sqrt{T} \quad (\mu = 0, \text{ i.i.d.})$$

Coherent risk measure ρ :

- Monotonicity: $R_1 \geq R_2 \Rightarrow \rho(R_1) \leq \rho(R_2)$
- Subadditivity: $\rho(R_1 + R_2) \leq \rho(R_1) + \rho(R_2)$
- Positive homogeneity: $\rho(\beta R) = \beta \rho(R)$, $\beta > 0$
- Translation invariance: $\rho(R + c) = \rho(R) - c$

VaR fails subadditivity for non-elliptical distributions $\Rightarrow$ not coherent. ES is always coherent.

Calculating VaR

Historical simulation: with n scenarios, 99% VaR = (0.01 n)th worst loss; ES = average of losses worse than VaR.

Linear derivative: $VaR_P = \delta \times VaR_f$

Delta-normal: $\Delta P = \sum \delta_i \Delta x_i$, $\sigma_P^2 = \sum_i \sum_j \delta_i \delta_j \rho_{ij} \sigma_i \sigma_j$

$$VaR = z \sigma_P - \mu_P$$

Option (delta-normal): $VaR_{\text{option}} = |\Delta| \times VaR_{\text{stock}} = |\Delta| S z \sigma$

Delta-gamma: $\Delta P \approx \delta \Delta S + \frac{1}{2} \gamma (\Delta S)^2$

Monte Carlo: 500 trials, 99% VaR = 5th worst loss, ES = average of 4 worst. Delta-normal understates tail for non-linear products.

Volatility: EWMA & GARCH

$$u_i = \ln \frac{S_i}{S_{i-1}} \quad (\text{or } \frac{S_i - S_{i-1}}{S_{i-1}}), \quad \sigma_n^2 = \frac{1}{m} \sum_{i=1}^m u_{n-i}^2$$

EWMA (RiskMetrics $\lambda = 0.94$ daily, 0.97 monthly):

$$\sigma_n^2 = \lambda \sigma_{n-1}^2 + (1 - \lambda) u_{n-1}^2$$

GARCH(1,1):

$$\sigma_n^2 = \omega + \alpha u_{n-1}^2 + \beta \sigma_{n-1}^2, \quad \omega = \gamma V_L, \quad \alpha + \beta + \gamma = 1$$

$$V_L = \frac{\omega}{1 - \alpha - \beta} \quad (\text{stable only if } \alpha + \beta < 1)$$

Mean reversion / forecast t days ahead:

$$E[\sigma_{n+t}^2] = V_L + (\alpha + \beta)^t (\sigma_n^2 - V_L)$$

EWMA covariance & correlation:

$$\text{cov}_n = \lambda \text{cov}_{n-1} + (1 - \lambda) x_{n-1} y_{n-1}, \quad \rho_n = \frac{\text{cov}_n}{\sigma_{x,n} \sigma_{y,n}}$$

EWMA = GARCH with $\omega = 0$, $\alpha = 1 - \lambda$, $\beta = \lambda$ (no mean reversion). Implied vol (e.g. VIX) is forward-looking.

Default probabilities & hazard rates

$$\text{Survival to } t = e^{-\tilde{\lambda}t}, \quad Q(t) = 1 - e^{-\tilde{\lambda}t}$$

$$P(\text{default in } [t_1, t_2]) = e^{-\tilde{\lambda}t_1} - e^{-\tilde{\lambda}t_2}$$

$$P(\text{default in yr } t \mid \text{survived to } t-1) = \frac{\text{Unconditional PD in yr } t}{1 - \text{Cumulative PD to } t-1}$$

$$\tilde{\lambda} = -\frac{1}{t} \ln[1 - Q(t)]$$

$$EL = PD \times LGD = PD \times (1 - RR)$$

Transition matrix: multi-year PD = matrix multiplication (sum over all paths). Senior secured RR $\approx$ 50%. PD and RR are negatively correlated.

Credit risk portfolio models

Single loan (face L):

$$E(\text{Loss}) = PD \times L(1 - RR), \quad \sigma_i = \sqrt{PD - PD^2} L(1 - RR)$$

Portfolio of n identical loans (correlation ρ):

$$\sigma_P = \sigma_i \sqrt{n + n(n-1)\rho}, \quad \frac{\sigma_P}{nL} = \frac{\sigma_i \sqrt{1 + (n-1)\rho}}{L\sqrt{n}}$$

Vasicek worst-case default rate (confidence X):

$$WCDR = N \left[\frac{N^{-1}(PD) + \sqrt{\rho} N^{-1}(X)}{\sqrt{1 - \rho}} \right]$$

$$\text{Capital (UL)} = EAD \times LGD \times (WCDR - PD)$$

Basel IRB uses $X = 99.9\%$, 1-year horizon. Company defaults if $U_i \leq N^{-1}(PD)$.

Euler's theorem: $\sigma_P = \sum_i x_i \frac{\partial \sigma_P}{\partial x_i}$ (risk contributions add to total).

Operational risk

Basic indicator: capital = 15% $\times$ avg. annual gross income (3 yrs)

Gross income = interest earned + non-interest income – interest paid.

Standardised: $\sum \beta_j \times GI_j$ (3-yr average) over 8 business lines:

18%: corporate finance, trading & sales, payment & settlement; 15%: commercial banking, agency services; 12%: retail banking, asset management, retail brokerage.

AMA: capital = 99.9th percentile (1 yr) – expected loss

SMA loss component: $LC = 7X + 7Y + 5Z$ (X all losses, $Y >$ EUR 10m, $Z >$ EUR 100m; 10-yr averages). Capital = $BIC \times ILM$.

Frequency: Poisson, $P(n) = \frac{e^{-\lambda} \lambda^n}{n!}$

Severity (lognormal), loss mean m , s.d. s :

$$w = (s/m)^2, \quad \sigma_{\ln}^2 = \ln(1 + w), \quad \mu_{\ln} = \ln m - \frac{1}{2} \sigma_{\ln}^2$$

Scale adjustment: $\text{Loss}_B = \text{Loss}_A \left(\frac{\text{Rev}_B}{\text{Rev}_A} \right)^{0.23}$

Power law: $P(v > x) = Kx^{-\alpha}$

Discount factors & bond pricing

$$P = \sum_t \frac{C}{2} d(t) + 100 d(T), \quad d(t) = \frac{1}{(1 + r_t/2)^{2t}}$$

Par rate (semiannual): $\frac{C}{2} \sum_t d(t) + d(T) = 1 \Rightarrow C = \frac{2[1 - d(T)]}{\sum_t d(t)}$

Forward rates: $(1 + r_T/2)^{2T} = \prod (1 + f_i/2)$ (spot = geometric average of forwards)

YTM: single y such that $P = \sum \frac{c/2}{(1 + y/2)^t} + \frac{100}{(1 + y/2)^{2T}}$

Annuity: $PV = \frac{C}{y} \left[1 - \frac{1}{(1 + y)^T} \right]$ **Perpetuity:** $PV = \frac{C}{y}$

Realised gross return: $\frac{P_{\text{end}} + \text{Coupons (incl. reinvestment)}}{P_{\text{begin}}} - 1$; net return

subtracts financing cost.

Spread: constant s added to all spot/forward rates so model price = market price.

P&L decomposition: carry-roll-down + rate changes + spread change. Law of one price: identical cash flows $\Rightarrow$ identical price (else arbitrage via replicating portfolio). Coupon $>$ YTM $\Rightarrow$ premium bond.

DV01, duration & convexity (VRM)

$$DV01 = -\frac{\Delta P}{10,000 \times \Delta y}$$

$$D_{\text{eff}} = \frac{P_- - P_+}{2P_0 \Delta y}, \quad C_{\text{eff}} = \frac{P_+ + P_- - 2P_0}{P_0 (\Delta y)^2}$$

$$\frac{\Delta P}{P} \approx -D \Delta y + \frac{1}{2} C (\Delta y)^2$$

$$DV01 = \frac{D \times P}{10,000}$$

Hedge (DV01-neutral): $F_H = -F_P \times \frac{DV01_P}{DV01_H}$ (DV01 per 100 face)

Portfolio: $D_P = \sum w_i D_i$, $C_P = \sum w_i C_i$ (value weights); $DV01_P = \sum DV01_i$

Barbell vs. bullet with equal cost & duration: barbell has more convexity.

Non-parallel shifts

$KR01_k = -\Delta P$ (price change for a 1bp rise in key rate k)

$$\sum_k KR01_k = DV01$$

$$\text{Key-rate duration}_k = \frac{KR01_k \times 10,000}{P}$$

Hedge: choose hedge amounts so that each KR01 of the hedge offsets the portfolio's KR01 (solve simultaneously, shortest key rate first). PCA: level, slope, curvature explain most variation.

Binomial trees

$$u = e^{\sigma\sqrt{\Delta t}}, \quad d = \frac{1}{u}$$

$$\pi_u = \frac{a-d}{u-d}, \quad \pi_d = 1 - \pi_u$$

$a = e^{r\Delta t}$ (stock); $e^{(r-q)\Delta t}$ (index, yield q); $e^{(r-r_f)\Delta t}$ (currency); 1 (futures).

$$f_0 = e^{-r\Delta t} [\pi_u f_u + \pi_d f_d]$$

$$\Delta = \frac{f_u - f_d}{S_0 u - S_0 d}$$

American: at each node take max(exercise value, continuation value). More steps $\Rightarrow$ converges to BSM.

Black–Scholes–Merton

$$c = S_0 N(d_1) - X e^{-rT} N(d_2)$$

$$p = X e^{-rT} N(-d_2) - S_0 N(-d_1)$$

$$d_1 = \frac{\ln(S_0/X) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}, \quad d_2 = d_1 - \sigma\sqrt{T}$$

Dividends: use $S_0 - PV(D)$, or $S_0 e^{-qT}$ for yield q (currency: $q = r_f$).

Black (futures): $c = e^{-rT} [F_0 N(d_1) - X N(d_2)]$, $d_1 = \frac{\ln(F_0/X) + \sigma^2 T/2}{\sigma\sqrt{T}}$

Lognormal stock price:

$$\ln S_T \sim N \left[\ln S_0 + \left(\mu - \frac{\sigma^2}{2} \right) T, \sigma^2 T \right], \quad E(S_T) = S_0 e^{\mu T}$$

Expected continuously compounded return $= \mu - \sigma^2/2$. Historical vol: $\hat{\sigma} = s/\sqrt{\tau}$ (τ = interval in years).

Warrants: value $= \frac{N}{N+M} \times \text{BSM call on } \left(S_0 + \frac{M}{N} W \right)$ (N shares, M warrants)

The Greeks

Greek	Call	Put
Delta Δ	$N(d_1)$	$N(d_1) - 1$
Gamma Γ	$\frac{N'(d_1)}{S_0 \sigma \sqrt{T}}$ (same for call & put)	
Vega ν	$S_0 \sqrt{T} N'(d_1)$ (same)	
Rho	$X T e^{-rT} N(d_2)$	$-X T e^{-rT} N(-d_2)$

$N'(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$. With yield q : call $\Delta = e^{-qT} N(d_1)$. Forward $\Delta = 1$ (no income); futures $\Delta = e^{rT}$.

Delta hedge: shares to trade $= -(\text{portfolio delta})$; e.g. short n calls $\Rightarrow$ buy $n \times \Delta$ shares.

Gamma-neutral: #options to add $= -\frac{\Gamma_P}{\Gamma_{\text{option}}}$, then re-hedge delta with stock.

Portfolio Greek $= \sum n_i \times \text{Greek}_i$

BSM PDE link:

$$\Theta + rS\Delta + \frac{1}{2}\sigma^2 S^2 \Gamma = r\Pi$$

Delta-neutral: $\Theta + \frac{1}{2}\sigma^2 S^2 \Gamma = r\Pi$. Long options: $\Gamma > 0$, $\Theta < 0$ (usually). Gamma & vega peak ATM.