

FRM® PART I · BOOK 4

Valuation & Risk Models

Detailed Formula Sheet

Every formula with variable definitions, intuition, worked examples and exam traps · Prepared by **RBei Classes**, Noida

What this book tests. Valuation & Risk Models carries about 30% of the Part I exam — the joint-largest book, and the most calculation-heavy. It has four blocks: **risk measurement** (VaR, expected shortfall, volatility), **credit, country, operational risk & stress testing**, **fixed-income valuation** (discounting, spot/forward/par rates, yields, DV01, duration, convexity, key rates) and **option valuation** (binomial trees, Black–Scholes–Merton, Greeks).

How to read this sheet. Maroon boxes = formulas Green boxes = fully worked examples (every step verified) Orange boxes = exam traps Blue-grey boxes = must-know facts.

Sources. Chapter n of this sheet = GARP *Valuation and Risk Models* Chapter n = Schweser Book 4 Reading 46 + n (e.g. Chapter 1 = Reading 47). **Notation.** μ = mean; σ = standard deviation (volatility); z = standard-normal critical value; $N(\cdot)$ = cumulative normal; $N'(\cdot)$ = normal density; V or P = value/price; y = yield; r = continuously compounded risk-free rate; T = years.

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1. Measures of Financial Risk

Mean–variance framework, VaR, expected shortfall, coherent risk measures

1.1 Mean–variance framework

$$\mu_P = \sum_i w_i \mu_i \quad \sigma_P^2 = \sum_i \sum_j w_i w_j \rho_{ij} \sigma_i \sigma_j \quad \text{two assets: } \sigma_P^2 = w_1^2 \sigma_1^2 + w_2^2 \sigma_2^2 + 2w_1 w_2 \rho \sigma_1 \sigma_2 \quad z = \frac{x - \mu}{\sigma}$$

Valid only for **elliptical** (e.g. normal) distributions. The efficient frontier dominates all other risky portfolios; with a risk-free asset the best portfolios lie on the straight line from R_f through the market portfolio M . Real equity returns are **more peaked with fatter tails** than the normal.

1.2 Value at risk (VaR) and expected shortfall (ES) under normality

$$\text{VaR} = (z\sigma - \mu) \times V$$

$$\text{ES} = \left(\frac{\sigma e^{-z^2/2}}{\sqrt{2\pi}(1-X)} - \mu \right) \times V = \left(\sigma \frac{N'(z)}{1-X} - \mu \right) \times V$$

WHERE X = confidence level (e.g. 95%); z = its one-tailed critical value; V = portfolio value; μ, σ = return mean and s.d. over the horizon.

$$\text{VaR}_T = \text{VaR}_1 \sqrt{T} \quad (\text{only if } \mu = 0 \text{ and returns are i.i.d.; if } \mu > 0, \text{ VaR grows more slowly})$$

Confidence X	90%	95%	97.5%	99%
VaR multiplier z	1.282	1.645	1.960	2.326
ES multiplier $N'(z)/(1-X)$	1.755	2.063	2.338	2.665

VaR = the loss that will not be exceeded with probability X (a quantile) — it says nothing about *how big* losses beyond it are. ES (= conditional VaR = expected tail loss) = the *average* loss given that the loss exceeds VaR, so $\text{ES} > \text{VaR}$.

Worked example — VaR and ES of a USD 100m portfolio

1-week $\mu = 0.00188$, $\sigma = 0.0125$, 95% (Schweser uses $z = 1.65$): $\text{VaR} = (1.65 \times 0.0125 - 0.00188) \times 100\text{m} = 0.018745 \times 100\text{m} = \text{USD } 1,874,500$.

ES with $\mu = 0$: $0.0125 \times 2.063 \times 100\text{m} = \text{USD } 2,577,770$ ($\approx 2.58\text{m}$). Check: ES/VaR ratio for $\mu = 0$ is $2.063/1.645 = 1.25$ ✓

Coherent risk measures — four properties

Monotonicity	$R_1 \geq R_2 \Rightarrow \rho(R_1) \leq \rho(R_2)$ — better returns in every state means less risk.
Subadditivity	$\rho(R_1 + R_2) \leq \rho(R_1) + \rho(R_2)$ — diversification never increases risk. The most important property.
Positive homogeneity	$\rho(\beta R) = \beta \rho(R)$ for $\beta > 0$ — doubling the position doubles the risk (needs a liquid market).
Translation invariance	$\rho(c + R) = \rho(R) - c$ — adding cash c reduces risk by c .
VaR vs. ES	VaR is not coherent: it can fail subadditivity for non-elliptical distributions. ES is coherent, has a convex risk surface (better for optimisation) and measures tail size.

2. Calculating & Applying VaR

Linear vs. non-linear derivatives, historical simulation, delta-normal, Monte Carlo, stress testing, worst-case scenario

2.1 Linear derivatives, delta-normal and delta-gamma

$\text{VaR}_P = \delta \times \text{VaR}_f$ forward value = $S - PV(K)$ (linear) index futures: $\text{VaR}(F) = \text{multiplier} \times \text{VaR}(\text{index})$

$$\Delta P = \sum_i \delta_i \Delta x_i \quad \sigma_{\Delta P}^2 = \sum_i \sum_j \delta_i \delta_j \rho_{ij} \sigma_i \sigma_j \quad \boxed{\text{VaR} = z \sigma_{\Delta P} - \mu_{\Delta P}} \quad \text{ES} = \sigma_{\Delta P} \frac{N'(z)}{1-X} - \mu_{\Delta P}$$

$$\boxed{\Delta P \approx \delta \Delta S + \frac{1}{2} \gamma (\Delta S)^2} \quad (\text{delta-gamma: adds curvature for options})$$

δ = sensitivity of the derivative to the risk factor; Δx_i = % change for prices (equities, commodities) or actual change for rates & spreads. A **linear** derivative has constant delta (not necessarily 1). Missing term-structure points are found by **linear interpolation** (6-month 2.2%, 1-year 2.8% $\Rightarrow$ 9-month 2.5%).

2.2 Simulation approaches

Historical simulation: revalue today's portfolio under each of n past daily changes, rank losses. With 300 scenarios, 99% VaR = **3rd-worst** loss ($300 \times 1\%$); ES = average of the losses *worse* than VaR.

Monte Carlo: (1) value portfolio today; (2) sample Δx_i from a multivariate distribution; (3) update risk factors; (4) revalue; (5) loss = today's value – new value; (6) repeat thousands of times. 500 trials: 99% VaR = 5th-worst; ES = average of the 4 worst (Schweser convention). Scale: $\text{VaR}_T = \text{VaR}_1 \sqrt{T}$. Historical: no distribution assumed, but limited to history. Monte Carlo: any distribution and correlation, handles non-linear products, but **slow and computationally intensive**.

Worked example — Historical and delta-normal VaR (illustrative numbers)

Historical: 300 daily scenarios; three worst losses are 16.0m, 13.0m, 11.0m. 99% VaR = 3rd worst = **11.0m**; ES = $(16.0 + 13.0)/2 = 14.5\text{m}$.

Delta-normal: long 1,000 calls, $\delta = 0.6$, stock 100, daily $\sigma = 2\%$. Delta exposure = $0.6 \times 1,000 \times 100 = 60,000$; $\sigma_{\Delta P} = 60,000 \times 0.02 = 1,200$; 99% VaR = $2.326 \times 1,200 = \text{2,791}$.

EXAM TRAP: The delta-normal method treats an option as linear: for a **long** option it *overstates* losses (ignores positive gamma); for a short option it *understates* them. Delta works better for deep in-/out-of-the-money options (low gamma) than at-the-money ones. Non-linear products: options, Asian and barrier options, MBS, butterfly spreads.

EXAM TRAP: In a crisis, volatilities and correlations rise together (**contagion**) — diversification disappears when needed most. Stress testing, stressed correlations and the **worst-case scenario** (distribution of the worst outcome over a period) supplement VaR — they do not replace it.

3. Measuring & Monitoring Volatility

Fat tails, regimes, parametric vs. non-parametric, EWMA, GARCH(1,1), mean reversion, implied volatility, correlation

3.1 Historical volatility

$$u_i = \ln \frac{S_i}{S_{i-1}} \text{ or } \frac{S_i - S_{i-1}}{S_{i-1}} \quad \sigma^2 = \frac{1}{m-1} \sum_{i=1}^m (u_i - \bar{u})^2 \quad \text{MLE with } \bar{u} = 0: \sigma_n^2 = \frac{1}{m} \sum_{i=1}^m u_{n-i}^2$$

$$\sigma_{\text{annual}} = \sigma_{\text{daily}} \sqrt{252} \quad \sigma_T = \sigma_{\text{daily}} \sqrt{T} \quad \text{general weights: } \sigma_n^2 = \sum_i \alpha_i u_{n-i}^2, \quad \sum \alpha_i = 1$$

3.2 EWMA and GARCH(1,1)

$$\text{EWMA: } \sigma_n^2 = \lambda \sigma_{n-1}^2 + (1 - \lambda) u_{n-1}^2$$

RiskMetrics: $\lambda = 0.94$ (daily), 0.97 (monthly)

$$\text{GARCH(1,1): } \sigma_n^2 = \omega + \alpha u_{n-1}^2 + \beta \sigma_{n-1}^2$$

$$\omega = \gamma V_L, \quad \gamma = 1 - \alpha - \beta,$$

$$V_L = \frac{\omega}{1 - \alpha - \beta}$$

$$\text{EWMA covariance: } \text{cov}_n = \lambda \text{cov}_{n-1} + (1 - \lambda) x_{n-1} y_{n-1} \quad \rho_n = \frac{\text{cov}_n}{\sigma_{x,n} \sigma_{y,n}}$$

EWMA = GARCH with $\omega = 0$, $\alpha = 1 - \lambda$, $\beta = \lambda$. Higher $\lambda \Rightarrow$ slower reaction to new returns. GARCH needs $\alpha + \beta < 1$; its V_L term pulls variance back to the long-run level (**mean reversion**); EWMA has **no** pull. Both capture volatility clustering.

Worked example — EWMA, GARCH and correlation updates

EWMA: $\lambda = 0.94$, $\sigma_{n-1} = 1\%$, $u = 2\%$: $\sigma_n^2 = 0.94(0.0001) + 0.06(0.0004) = 0.000118 \Rightarrow \sigma_n = \mathbf{1.086\%}$.

GARCH: $\omega = 0.000003$, $\alpha = 0.04$, $\beta = 0.92$: $\sigma_n^2 = 0.000003 + 0.000016 + 0.000092 = 0.000111 \Rightarrow \mathbf{1.054\%}$; $V_L = 0.000003/0.04 = 0.0000075 \Rightarrow$ long-run vol **0.866%**.

Correlation: $\rho = 0.3$, $\sigma_x = 1\%$, $\sigma_y = 2\% \Rightarrow \text{cov} = 0.000006$; returns $x = 2\%$, $y = 4\%$, $\lambda = 0.92$: $\text{cov}_n = 0.92(0.000006) + 0.08(0.02)(0.04) = 0.0001192$; with new vols 1.11% and 2.23% : $\rho_n = 0.0001192 / (0.0111 \times 0.0223) = \mathbf{0.482}$.

Horizon: daily $1.5\% \Rightarrow$ 30-day $1.5\% \sqrt{30} = 8.2\%$; if mean reversion lifts expected daily vol to 1.8% : $1.8\% \sqrt{30} = \mathbf{9.9\%}$.

Must-know facts — volatility

Deviations from normal	(1) fat tails , (2) skew (e.g. mixing two distributions), (3) unstable parameters. Fat tails in the <i>unconditional</i> distribution usually come from time-varying volatility, even if each <i>conditional</i> distribution is normal.
Regime switching	Volatility jumps abruptly between states (e.g. after a surprise central-bank announcement).
Parametric vs. non-parametric	Parametric (e.g. delta-normal, EWMA) uses data efficiently but assumes a distribution. Non-parametric (historical simulation, MDE) needs no distribution but needs lots of data. MDE weights past periods by similarity of state variables ; risk of overfitting.
Implied volatility	Forward-looking, backed out from option prices (usually annual; daily = $\sigma / \sqrt{252}$, e.g. $18\% \Rightarrow 1.13\%$). VIX = 30-day S&P 500 implied vol (“fear index”; India: India VIX on NIFTY options).

4. External & Internal Credit Ratings

Rating scales, default probabilities, hazard rates, recovery, transition matrices, TTC vs. PIT

4.1 Default probabilities and hazard rates

$$\text{Conditional PD}_t = \frac{\text{Unconditional PD}_t}{\text{Survival to start of } t}$$

$$\text{Unconditional PD in year } t = \text{Cum PD}_t - \text{Cum PD}_{t-1}$$

$$\text{Cum PD}(t) = 1 - e^{-\bar{\lambda}t}$$

$$\text{Survival} = e^{-\bar{\lambda}t}$$

$$\text{PD between } t_1, t_2 = e^{-\bar{\lambda}t_1} - e^{-\bar{\lambda}t_2} \quad \bar{\lambda} = -\frac{\ln(1 - \text{Cum PD})}{t}$$

$$\text{EL} = \text{PD} \times \text{LGD} = \text{PD} \times (1 - \text{RR})$$

$$\text{credit spread (actuarial)} \approx \frac{\text{EL}}{\text{years}}$$

λ = hazard rate (default intensity); RR = recovery rate = bond value just after default as % of face (senior secured $\approx 50\%$, junior $\approx 25\%$). Default rates and recovery rates are **negatively** correlated.

Worked example — Hazard rates, EL and rating migration

Constant $\lambda = 2\%$: default by year 4 = $1 - e^{-0.08} = 7.69\%$; unconditional PD in year 5 = $e^{-0.08} - e^{-0.10} = 1.83\%$; conditional on surviving 4 years = $1.83 / (1 - 0.0769) = 1.98\%$.

B-rated: cumulative PD 8.56% (2 yr), 12.66% (3 yr) $\Rightarrow$ year-3 unconditional = 4.10%; conditional = $4.10 / 91.44 = 4.48\%$; average hazard = $-\ln(1 - 0.1266) / 3 = 4.51\%$.

4-yr PD 15.87%, RR 20%: EL = $15.87\% \times 0.80 = 12.70\% \Rightarrow \approx 318 \text{ bp}$ per year.

Migration: B stays B 85%, $\rightarrow$ A 5%, $\rightarrow$ C 5%, defaults 5%; year-2 PDs from A/B/C = 0/5/15%: 2-yr PD = $5\% + 0.85(5\%) + 0.05(0) + 0.05(15\%) = 10\%$.

Must-know facts — ratings

Scales	Investment grade: S&P/Fitch AAA–BBB– , Moody's Aaa–Baa3 . Below = speculative / high yield. Indian agencies: CRISIL, ICRA, CARE, India Ratings.
PD behaviour	For investment grade, annual PD <i>rises</i> with time; for the lowest grades, surviving early years <i>lowers</i> later conditional PD.
Outlook vs. watchlist	Outlook: medium-term direction (positive, negative, stable, developing). Watchlist: short-term (≤ 3 months) likely change.
TTC vs. PIT	Agencies rate through-the-cycle (stable, ignores the business cycle). Point-in-time = current PD; falls in recessions. Internal ratings tend to be PIT.
Transition matrix	One-year probabilities of moving between grades; downgrades tend to be followed by downgrades (momentum). Multi-year matrices $\approx$ matrix multiplied by itself.
Market reaction	Downgrades (esp. investment grade $\rightarrow$ junk) move stock and bond prices; upgrades much less. CDS spreads tend to <i>anticipate</i> rating changes.
Alternatives	Kamakura, Moody's KMV: models using equity value, equity volatility and debt — point-in-time. Internal ratings needed where no external rating exists, for IFRS 9/CECL, and for IRB capital.
Structured products	2007–09: over-optimistic default correlations, products engineered to ratings, conflicts of interest $\Rightarrow$ now more oversight.

5. Country Risk

Sources of country risk, composite measures, sovereign default, sovereign spreads & CDS

Must-know facts — country risk

Sources	Economic growth (level and volatility; diversification vs. reliance on one commodity), political risk (continuity, corruption, violence, expropriation; democracy vs. autocracy), legal risk (property rights, speed and fairness of courts), and the economic structure .
Composite measures	Political Risk Services (PRS) , Euromoney , The Economist , World Bank governance indicators — combine many factors into one score.
Foreign vs. local debt	Foreign-currency default is more common (can't print foreign currency). Local-currency default also happens (to avoid inflation, due to gold standard / shared currency like the euro, or currency debasement). Local rating is usually $\geq$ foreign rating.
Consequences	Lost market access, higher borrowing costs, GDP decline, capital flight, banking-system stress, political change.
Default-risk factors	Debt-to-GDP (not debt-to-equity), social-security/pension commitments, stability of the tax base , political risk, independence of the central bank, implicit guarantees (e.g. EU support).
Sovereign spread	Sovereign yield – riskless yield in the <i>same currency and maturity</i> , e.g. USD-bond yield 8% – US Treasury 2% = 6% . More dynamic than ratings; reacts faster.
Sovereign CDS	Protection against sovereign default; informative but can be illiquid, has counterparty risk and was blamed on speculators (Greece 2010; EU banned uncovered sovereign CDS purchases).

6. Measuring Credit Risk

EL & UL, economic vs. regulatory capital, loan-loss s.d., Gaussian copula, Vasicek, CreditMetrics, Euler

6.1 Expected loss, unexpected loss and capital

$$\text{EL} = \text{EAD} \times \text{PD} \times \text{LGD}$$

$$\text{LGD} = 1 - \text{RR}$$

$$\text{Capital} = \text{VaR}_X - \text{EL} \quad (1 \text{ year}, X = 99.9\% \text{ for IRB})$$

EAD = exposure at default; PD = probability of default; LGD = loss given default. EL is priced/provisioned for; capital covers **unexpected loss** (actual – expected). A AA bank targets $\approx 99.98\%$ ($\text{PD} \leq 0.02\%$). **Economic** capital = bank's own estimate; **regulatory** capital = Basel requirement (standardised or IRB).

6.2 Standard deviation of loan-portfolio losses (binomial)

$$\sigma_i = \sqrt{\text{PD}_i - \text{PD}_i^2} \times L_i(1 - \text{RR}_i) \quad \sigma_P^2 = \sum_i \sum_j \rho_{ij} \sigma_i \sigma_j$$

$$\text{Identical loans: } \sigma_P = \sigma \sqrt{n + n(n-1)\rho}$$

$$\frac{\sigma_P}{nL} = \frac{\sigma \sqrt{1 + (n-1)\rho}}{L\sqrt{n}}$$

6.3 One-factor Gaussian copula and the Vasicek model

$$U_i = a_i F + \sqrt{1 - a_i^2} Z_i \quad \rho_{ij} = a_i a_j \quad \text{default if } U_i \leq N^{-1}(\text{PD})$$

$$\text{WCDR}_X = N \left[\frac{N^{-1}(\text{PD}) + \sqrt{\rho} N^{-1}(X)}{\sqrt{1 - \rho}} \right]$$

$$\text{UL (capital)} = \text{EAD} \times \text{LGD} \times (\text{WCDR} - \text{PD})$$

WCDR = worst-case (99.9th percentile) default rate; ρ = copula correlation. WCDR rises with both PD and ρ . The copula maps any marginal distribution to the standard normal **percentile-to-percentile** (1st percentile $\rightarrow -2.326$, 5th $\rightarrow -1.645$). **CreditMetrics** (economic capital) also captures losses from **rating migration**, via Monte Carlo with a Gaussian copula (correlations from equity returns).

6.4 Euler's theorem — risk contributions

$$F(\lambda x_1, \dots, \lambda x_n) = \lambda F(x_1, \dots, x_n) \Rightarrow F = \sum_i Q_i, \quad Q_i = x_i \frac{\partial F}{\partial x_i} \approx \frac{\Delta F}{\Delta x_i / x_i}$$

For σ_P : $Q_i = \frac{\sigma_i^2 + \sum_{j \neq i} \rho_{ij} \sigma_i \sigma_j}{\sigma_P}$. Contributions add up exactly to total risk — used to allocate capital to loans or business units.

Worked example — Loss s.d., Vasicek capital and Euler

10,000 loans of 1, PD 0.5%, RR 30%, $\rho = 0.2$: $\sigma_i = \sqrt{0.005 - 0.000025 \times 0.7} = \mathbf{0.04937}$; portfolio = $0.04937 \sqrt{(1 + 9,999 \times 0.2)/10,000} = 0.04937 \times 0.4473 = \mathbf{2.21\%}$ of size.

Quiz check: PD 1.1%, RR 40%, $n = 50,000 \Rightarrow \sigma_i = 0.06258$, portfolio 0.027988 ✓

Vasicek: PD 1%, $\rho = 0.2$: WCDR = $N[(-2.326 + \sqrt{0.2 \times 3.090})/\sqrt{0.8}] = N(-1.056) = \mathbf{14.55\%}$. Loan 100m, LGD 40%: EL = 0.40m; capital = $100 \times 0.4 \times (0.1455 - 0.01) = \mathbf{5.42m}$.

Euler (illustrative matrix): $\sigma_{1,2,3} = 1.2, 0.8, 0.8$; $\rho_{23} = 0.6$, others 0: $\sigma_P = \sqrt{1.44 + 0.64 + 0.64 + 2(0.6)(0.64)} = \mathbf{1.868}$; $Q_1 = 1.44/1.868 = 0.771$; $Q_2 = Q_3 = (0.64 + 0.384)/1.868 = 0.548$; sum = 1.868 ✓ (Schweser's 1% bump gives $0.7733 + 0.5492 + 0.5492 \approx 1.87$).

EXAM TRAP: Defaults are **correlated** (clustered in recessions — 1990, 2001, 2009) $\Rightarrow$ losses do not diversify away. EL grows with exposure; UL can exceed EL by a wide margin. Derivatives are harder than loans: EAD = current exposure **plus an add-on**, and **netting** means capital is calculated per counterparty. Regulators want through-the-cycle PDs; banks often prefer point-in-time; **wrong-way risk** (exposure rises as counterparty weakens) is hard to model.

7. Operational Risk

Basel categories, BIA, standardised approach, AMA, SMA, loss distribution, scaling, power law, insurance

7.1 Regulatory capital approaches

$$\begin{aligned} \text{BIA: } K &= 15\% \times \overline{GI}_{3 \text{ yrs}} & GI &= \text{interest earned} - \text{interest paid} + \text{non-interest income} \\ \text{Standardised: } K &= \sum_{8 \text{ lines}} \beta_j \times \overline{GI}_j & \text{AMA: } K &= 99.9\text{th percentile of 1-yr loss} - \text{EL} \quad (56 = 8 \times 7 \text{ cells}) \end{aligned}$$

$$\text{SMA: } LC = 7X + 7Y + 5Z \quad K = \text{BIC} \times \text{ILM}$$

SMA (Basel, 2016, replaces all three): BI = business indicator (like gross income, but trading losses and operating expenses *increase* it); BIC = BI component (a piecewise % of BI); LC = loss component, where X, Y, Z = 10-year average annual losses (all losses; losses > EUR 10m; losses > EUR 100m); ILM = internal loss multiplier from LC vs. BIC. For an average bank $LC = \text{BIC}$ and $ILM = 1$.

Standardised approach β (% of gross income)			
Corporate finance	18%	Commercial banking	15%
Trading & sales	18%	Agency services	15%
Payment & settlement	18%	Retail banking	12%
Asset management	12%	Retail brokerage	12%

7.2 Loss distribution approach

$$\text{Frequency (Poisson): } P(n) = \frac{e^{-\lambda} \lambda^n}{n!} \quad \text{Severity (lognormal): } \mu_{\ln} = \ln \frac{m}{\sqrt{1+w}}, \quad \sigma_{\ln}^2 = \ln(1+w), \quad w = \left(\frac{s}{m}\right)^2$$

$$\text{Loss}_B = \text{Loss}_A \left(\frac{\text{Revenue}_B}{\text{Revenue}_A} \right)^{0.23}$$

$$\text{Power law: } P(v > x) = Kx^{-\alpha}$$

λ = average number of losses per year; m, s = mean and s.d. of loss size. Monte Carlo: sample n from Poisson $\rightarrow$ sample n severities $\rightarrow$ sum $\rightarrow$ repeat thousands of times $\rightarrow$ read percentiles. UL = percentile – mean. Power law holds in the tail (top $\approx 5\%$); smaller α = fatter tail.

Worked example — Operational-risk calculations

BIA: gross income 100, 120, 140 $\Rightarrow K = 0.15 \times 120 = \mathbf{18}$.

Severity: mean 60m, s.d. 20m: $w = (20/60)^2 = 0.1111$; $\sigma_{\ln}^2 = \ln(1.1111) = \mathbf{0.105}$; $\mu_{\ln} = \ln(60/\sqrt{1.1111}) = \ln(56.92) = \mathbf{4.042}$ (Schweser rounds w to 0.11, giving 0.104).

Scaling: Bank Z loses 5m on revenue 1bn; Bank Y revenue 2bn: $5 \times 2^{0.23} = \mathbf{5.86m}$ (not 10m).

Frequency: 10 losses in 5 years $\Rightarrow \lambda = 2$: $P(0) = e^{-2} = 13.5\%$; $P(3) = e^{-2} 2^3 / 6 = \mathbf{18.0\%}$.

Must-know facts — operational risk

Definition	Risk of loss from inadequate or failed internal processes, people and systems or from external events (includes legal risk; excludes strategic and reputational risk in Basel's definition).
7 Basel categories	Internal fraud; external fraud; employment practices & workplace safety; clients, products & business practices; damage to physical assets; business disruption & system failures; execution, delivery & process management.
Large risks	Cyber risk, compliance risk, rogue trader risk (Barings, Société Générale; India: PNB–Nirav Modi LoU fraud).
Data issues	Internal data for frequency; internal + external for severity. Adjust for inflation and scale . Vendor data is biased to large losses $\Rightarrow$ use it for <i>relative</i> severity. Scenario analysis fills gaps for rare, severe events.
Forward-looking tools	Causal analysis; RCSA (managers self-assess — may hide out-of-control risks); KRIs (staff turnover, unfilled posts, failed trades — must be predictive and timely); training.
Capital allocation	Allocating op-risk capital to business units rewards better controls (subject to cost–benefit).
Insurance	Moral hazard (insured bank takes less care — fix with deductibles, co-insurance, limits); adverse selection (riskier banks buy more — fix by examining internal controls and pricing accordingly).

8. Stress Testing

Stressed VaR/ES, scenario design, reverse stress testing, regulatory tests, governance, Basel principles

Must-know facts — stress testing

Rationale	Forward-looking check on <i>capital and liquidity</i> under severe but plausible scenarios; complements VaR/ES, which rely on recent data.
Stressed VaR / ES	VaR/ES computed on data from a specific stressed period (e.g. 2008). <i>Pros</i> : conservative, forward-looking. <i>Cons</i> : does not respond to current conditions; no probability distribution; cannot be back-tested.
Scenario design	Horizon usually 3 months to 2 years . Historical (1987 crash, 1998 Russia, 2008 Lehman, 2020 COVID-19) or hypothetical. Stress core variables in detail and peripheral ones roughly. Capture knock-on effects (others' reactions, e.g. interbank lending freeze). Correlations rise in stress.
Ad hoc tests	Non-routine scenarios (e.g. an election result, a new regulation).
Reverse stress test	Start from failure and ask which scenarios would cause it; exposes hidden vulnerabilities.
Regulatory tests	US: CCAR (capital plans, qualitative) and DFAST (Dodd–Frank); EU: EBA tests; India: RBI stress tests in the Financial Stability Report.
Governance	Board : approves and challenges scenarios, is ultimately responsible. Senior management : implements, reports, acts on results. Policies & procedures, independent validation , and internal audit review the whole process.
Basel principles	Part of risk governance; firm-wide; cover all material risks; severe but plausible; used in capital and liquidity planning; results should lead to management action; regularly reviewed.

9. Pricing Conventions, Discounting & Arbitrage

Discount factors, T-bills, bootstrapping, law of one price, STRIPS, replication, clean vs. dirty, day counts

9.1 Discount factors and bond prices

$$\text{T-bill cash price} = 100 - \frac{n}{360} \times Q \quad \boxed{P = \sum_t \frac{c}{2} d(t) + 100d(T)} \quad d(t) = \frac{PV}{FV}$$

Q = quoted (annualised discount) price; n = days to maturity; $d(t)$ = value today of 1 paid at t . **Bootstrapping**: solve $d(0.5)$ from the shortest bond, then use it to solve $d(1)$, and so on. Mid-market discount factor = midpoint of bid and ask cash prices (80-day bill: bid 99.60, ask 99.65 $\Rightarrow d = 0.99625$).

9.2 Clean and dirty prices

$$\boxed{\text{Dirty (full, invoice)} = \text{Clean (flat, quoted)} + \text{AI}}$$

$$\text{AI} = \frac{c}{2} \times \frac{\text{days since last coupon}}{\text{days in coupon period}}$$

$$P_{\text{dirty}} = \sum_{i=0}^{N-1} \frac{c/2}{(1+y/2)^{i+w}} + \frac{100}{(1+y/2)^{N-1+w}}$$

$$w = \frac{\text{days to next coupon}}{\text{days in coupon period}}$$

Day count	Typical use
Actual/actual	US Treasury bonds
30/360	US corporate and municipal bonds
Actual/360	US money market (T-bills, SOFR)
Actual/365	Money market in Canada, Australia, New Zealand (also India's G-secs money market)

Worked example — Bootstrapping, arbitrage and accrued interest

Bootstrap: $d(1) = 0.9939$; 7.25% bond priced 105.98: $3.625(0.9939) + 103.625d(2) = 105.98 \Rightarrow d(2) = \mathbf{0.9880}$.

Arbitrage: $d(1) = 0.96154$, $d(2) = 0.85734$; a 2-yr 8% annual bond is worth $8(0.96154) + 108(0.85734) = 100.285$ but trades at 100 $\Rightarrow$ **buy** USD 1m of the bond, **short** 80,000 face of 1-yr zeros (+76,923) and 1.08m face of 2-yr zeros (+925,927): net today = 1,002,850 – 1,000,000 = **+USD 2,850**, with all future cash flows offsetting.

AI: 10% semiannual, 1,000 par, 90 days since coupon (30/360): $50 \times 90/180 = \mathbf{25}$. EUR 100,000, 3.5%, 75 days: $1,750 \times 75/180 = \mathbf{729.17}$.

Must-know facts — conventions

Law of one price	Identical cash flows must have identical prices; otherwise buy the cheap one, short the rich one.
STRIPS	Coupon bonds split into C-STRIPS (coupons; TINTs/INTs) and P-STRIPS (principal; TPs). C- and P-STRIPS maturing on the same date can trade at different prices (P-STRIPS often richer); transaction costs may stop arbitrage.
Replicating portfolio	Match the target bond's cash flows with other bonds, solving from the longest maturity backwards. If target is cheap: buy it, short the replicating portfolio.
Short selling	The short seller pays coupons/dividends to the lender; risks: price rises or the security is recalled.

10. Interest Rates

Compounding, spot, forward & par rates, FRAs, swaps, curve shapes, flatteners & steepeners

10.1 Compounding and equivalent rates

$$FV = A \left(1 + \frac{R_m}{m}\right)^{mn} \quad FV = Ae^{R_c n} \quad \left(1 + \frac{R_{m1}}{m_1}\right)^{m_1} = \left(1 + \frac{R_{m2}}{m_2}\right)^{m_2}$$

$$R_c = m \ln \left(1 + \frac{R_m}{m}\right)$$

$$R_m = m \left(e^{R_c/m} - 1\right)$$

10.2 Spot, forward and par rates

$$d(t) = \frac{1}{(1 + r(t)/2)^{2t}} \quad (\text{semiannual}) \quad d(t) = e^{-r(t)t} \quad (\text{continuous})$$

$$(1 + r_2)^2 = (1 + r_1)(1 + f_{1,1}) \quad \left(1 + \frac{r_{T+0.5}}{2}\right)^{2T+1} = \left(1 + \frac{r_T}{2}\right)^{2T} \left(1 + \frac{f}{2}\right) \quad F_{T_1, T_2} = \frac{R_2 T_2 - R_1 T_1}{T_2 - T_1}$$

$$\text{Par rate } p_T = \frac{2[1 - d(T)]}{A_T}, \quad A_T = \sum_t d(t) \quad \text{Bond with coupon } c: P = 1 + \frac{c - p_T}{2} A_T$$

$$\text{FRA value} = \text{Notional} \times (R_K - R_F) \times (T_2 - T_1) \times d(T_2) \quad (\text{to the receiver of fixed } R_K)$$

Upward-sloping curve: forward > spot > par; downward-sloping: forward < spot < par. **Swap rates are par rates**, so the swap curve gives par rates, spot rates and discount factors. A bond's price *rises* as it ages if the forward rate for its final period exceeds its coupon.

Worked example — Rates

6% semiannual = **5.93%** monthly = **5.91%** continuous ($2 \ln 1.03$); 5% continuous = $2(e^{0.025} - 1) = \mathbf{5.06\%}$ semiannual.
 Spot: 50 grows to 58 in 2 years $\Rightarrow r = (58/50)^{0.5} - 1 = \mathbf{7.70\%}$; $d(2) = 50/58 = 0.86207$. 3-yr 6.25% continuous: $d = e^{-0.1875} = \mathbf{0.8290}$.
 Forward: 1-yr 2%, 2-yr 3.5%: $f = 1.035^2/1.02 - 1 = \mathbf{5.02\%}$. Continuous 4.25% (3y), 4.40% (3.5y): $F = (4.40 \times 3.5 - 4.25 \times 3)/0.5 = \mathbf{5.30\%}$.
 Par: $d = 0.9840, 0.9662, 0.9427, 0.9201$; $A = 3.813$: $p = 2(1 - 0.9201)/3.813 = \mathbf{4.19\%}$. A 3.25% bond: $1 + (0.0325 - 0.0419)/2 \times 3.813 = \mathbf{0.9821}$ per 1 par.

Curve shapes and trades

Shapes & shifts	Normal (upward), flat, inverted. Parallel shift = all rates move equally; non-parallel = twists (slope) and butterfly (curvature) shifts.
Flattening	Bull flattener : long rates fall more than short. Bear flattener : short rates rise more than long.
Steepening	Bear steepener : long rates rise more than short. Bull steepener : short rates fall more than long.
Trades	Expect flattening : long long-maturity bonds, short short-maturity. Expect steepening : short long-maturity, long short-maturity (DV01-weighted).
Forward view	2-yr 3%, 3-yr 3.5% (continuous) $\Rightarrow$ year-3 forward 4.5%. If you think the year-3 rate will be < 4.5%: borrow for 2 years, invest for 3.
Swap	Pay-fixed/receive-floating: gains if rates rise; loses (makes net payments) if the floating rate falls below the fixed rate. Notional is never exchanged.

11. Bond Yields & Return Calculations

Realised returns, spreads, YTM, annuities & perpetuities, coupon effect, P&L decomposition, carry roll-down

11.1 Returns and spreads

$$R_{t-1,t} = \frac{BV_t + C_t - BV_{t-1}}{BV_{t-1}} \quad (\text{gross}) \quad \text{Net} = \text{Gross} - \text{financing cost per period}$$

$$\text{Multi-period: } (1 + R_{t,t+1})(1 + R_{t+1,t+2}) \cdots - 1 \quad \text{Spread } s: P = \frac{C}{1 + f_1 + s} + \frac{C + 100}{(1 + f_1 + s)(1 + f_2 + s)}$$

BV = bond value; C = coupon (reinvested coupons count at the reinvestment rate — **reinvestment risk**). The spread s added to every forward rate makes the model price equal the market price: positive $s \Rightarrow$ bond is cheap.

11.2 Yield to maturity, annuities and perpetuities

$$P = \sum_{t=1}^T \frac{C}{(1+y)^t} + \frac{F}{(1+y)^T} \quad \text{periodic: } P = \sum_{t=1}^{mT} \frac{C/m}{(1+y/m)^t} + \frac{F}{(1+y/m)^{mT}}$$

$$\text{Annuity: } PV = \frac{C}{y} \left[1 - \frac{1}{(1+y)^T} \right]$$

$$\text{Perpetuity: } PV = \frac{C}{y}$$

$$\text{Japanese simple yield} = \frac{c + \frac{100-P}{T}}{P}$$

YTM = the IRR; assumes all coupons are reinvested at the YTM and the bond is held to maturity. Coupon $>$ YTM $\Rightarrow$ premium; coupon $<$ YTM $\Rightarrow$ discount. **Coupon effect**: bonds with equal maturity but different coupons have different YTM; in an upward-sloping curve, *higher* coupon $\Rightarrow$ *lower* YTM. Lower coupon $\Rightarrow$ more interest-rate risk.

11.3 P&L decomposition

$$\text{Price change} = \text{Carry roll-down} + \text{Rate change} + \text{Spread change} \quad (+ \text{ financing, } + \text{ accrued interest if needed})$$

Carry roll-down: return if expectations are realised (coupon plus price move as the bond “rolls down” the curve). Three “no change” assumptions: **realised forwards** (forward rates become future spot rates; return = the one-period rate), **unchanged term structure**, **unchanged yields**.

Worked example — Returns, yields and P&L

Bought 105, now 112, coupon 2 (6 months): gross = $(112 + 2 - 105)/105 = 8.571\%$. Held 1 year, earlier coupon reinvested at 1% p.a.: $(112 + 2 + 2.01 - 105)/105 = 10.486\%$. Fully financed at 0.6% p.a. for 6 months: net = $8.571\% - 0.30\% = 8.271\%$.

YTM: 100 a year for 10 years, price 700 $\Rightarrow 7.07\%$; if paid semiannually for 5 years, periodic yield 7.07% $\Rightarrow$ YTM **14.14%**.

Annuity 100×10 yrs at 10% = **614.46**; perpetuity $1,000/0.10 = 10,000$. Japanese: 3% coupon, 6 yrs, price 98: $(3 + 2/6)/98 = 3.40\%$.

P&L: $101.88 + 2.00 - 102.65 = 1.23 = 0.85$ (carry roll-down) $+ 0.30$ (rates) $+ 0.08$ (spread); gross return = $1.23/102.65 = 1.198\%$.

Realised forwards: 2-yr 2% bond (100.785) with forwards 1.0%, 1.2%, 1.4%: price in 6 months = $\frac{1}{1.005} + \frac{1}{1.005 \times 1.006} + \frac{101}{1.005 \times 1.006 \times 1.007} = 101.188$. Carry at the 0.8% 6-month rate: $100.785 \times 0.004 = 0.403$.

12. DV01, Duration & Convexity

One-factor models, DV01 hedging, effective duration, convexity, portfolios, two-bond hedges, barbell vs. bullet

12.1 DV01 and hedge ratio

$$DV01 = -\frac{\Delta P}{10,000 \times \Delta y}$$

$$DV01 = \frac{D \times P}{10,000}$$

$$HR = \frac{DV01_{\text{position}}}{DV01_{\text{hedge}}}, \text{Face}_{\text{hedge}} = \text{Face}_{\text{position}} \times HR$$

DV01 = price change for a 1 bp rate change (positive for a normal long bond). Variants: yield-based DV01, **DVDZ/DPDZ** (spot-rate shift), **DVDF/DPDF** (forward-rate shift). Portfolio DV01 = **simple sum** of DV01s.

12.2 Duration and convexity

$$D = -\frac{1}{P} \frac{\Delta P}{\Delta y} = \frac{P_- - P_+}{2P_0 \Delta y}$$

$$C = \frac{P_+ + P_- - 2P_0}{P_0 (\Delta y)^2}$$

$$D_{\text{mod}} = \frac{D_{\text{Mac}}}{1 + y/m}$$

$$\frac{\Delta P}{P} \approx -D \Delta y + \frac{1}{2} C (\Delta y)^2$$

$$D_P = \sum w_i D_i, \quad C_P = \sum w_i C_i \quad (w_i = \text{value weights})$$

P_+, P_- = prices after rates rise/fall by Δy . Duration alone *overstates* price falls and *understates* price rises; convexity is always a **positive** adjustment for option-free bonds. Callable bonds: upside capped (negative convexity at low rates); simple approach: $D = (1 - p_{\text{call}})D_{\text{maturity}} + p_{\text{call}}D_{\text{call date}}$; best approach: revalue with the option for ± 1 bp.

12.3 Duration and convexity hedging; barbell

$$\text{Duration hedge: } P_H = -\frac{V D_V}{D_H}$$

$$\text{Two bonds: } V D_V + P_1 D_1 + P_2 D_2 = 0, \quad V C_V + P_1 C_1 + P_2 C_2 = 0$$

$$\text{Barbell weight in short bond: } w_S = \frac{D_{\text{target}} - D_L}{D_S - D_L} \quad (\text{same cost and duration as the bullet, higher convexity})$$

Worked example — DV01, duration, convexity and hedges

DV01: 100,000 face, 6% semiannual, 5 yrs, price 100,750; 10 bp fall $\Rightarrow$ 101,181.44: $DV01 = 431.44/10 = \mathbf{43.14}$ (10 bp rise: -429.24 — convexity).

Hedge: portfolio 1m with DV01 340, hedge bond DV01 285: face = $1\text{m} \times 340/285 = \mathbf{1,192,982}$.

Duration: $D = 3.28$, $P = 98,550$, rates -25 bp: $\Delta P = -3.28 \times 98,550 \times (-0.0025) = \mathbf{+808.11}$.

Convexity: par bond; $+50$ bp $\rightarrow$ 95.586, -50 bp $\rightarrow$ 104.701: $C = (95.586 + 104.701 - 200)/(100 \times 0.005^2) = \mathbf{114.8}$.

$D = 9.115$, $C = 114.8$, ± 150 bp: $+150$: $-13.673 + 1.292 = -12.381$; -150 : $13.673 + 1.292 = \mathbf{+14.964}$ (duration alone: ∓ 13.673).

Duration hedge: 1.7m, $D = 6.5$, hedge $D = 7.9$: $1.7 \times 6.5/7.9 = \mathbf{1.399\text{m short}}$ ($\approx 1.4\text{m}$).

Two-bond: 3m, $D = 6$, $C = 25$; bond 1 ($D = 7$, $C = 20$), bond 2 ($D = 5$, $C = 19$): $7P_1 + 5P_2 = -18$, $20P_1 + 19P_2 = -75 \Rightarrow P_1 = \mathbf{+1\text{m}}$, $P_2 = \mathbf{-5\text{m}}$.

EXAM TRAP: Duration = *percentage* sensitivity (investing); DV01 = *money* sensitivity (hedging and trading — positions of different sizes). Portfolio DV01 is a plain sum; portfolio duration and convexity are value-weighted averages. A barbell and a bullet with equal duration: the **barbell has higher convexity** (e.g. 116.1 vs. 59.8) and wins if rates are volatile; the bullet wins if the curve is stable.

13. Non-Parallel Shifts: PCA, Key Rates & Buckets

Principal components, key rate '01s and durations, key-rate hedging, partial '01s, forward-bucket '01s

13.1 Principal components analysis (PCA)

$$\text{Importance of factor } i = \frac{\sigma_i^2}{\sum_j \sigma_j^2} \quad \sigma_{\Delta P} = \sqrt{\sum_i (f_i \sigma_i)^2}$$

Factors are uncorrelated. σ_i = s.d. of factor score i ; f_i = portfolio value change per unit of factor i . Factor 1 $\approx$ level (shift), Factor 2 $\approx$ slope (twist), Factor 3 $\approx$ curvature (butterfly). The first 2–3 factors explain most of the variance.

13.2 Key rates, partial '01s and forward buckets

$$KR01_k = -\frac{\Delta V}{1 \text{ bp shift in key rate } k} \quad \sum_k KR01_k = DV01 \quad D_k = \frac{KR01_k}{V} \times 10,000$$

Hedge: choose positions x_j so that $\sum_j x_j KR01_{j,k} = -KR01_{P,k}$ for every key rate k

US key rates: **2-, 5-, 10- and 30-year par yields** (most liquid). A key-rate shift moves that rate by 1 bp and neighbouring rates by linearly declining amounts. **Partial '01s**: same idea for swap curves with more points. **Forward-bucket '01**: loss for a 1 bp rise in *all forward rates* in a bucket; the sum over buckets = DV01 to a parallel forward shift. Basel requires 10 key rates: 3m, 6m, 1, 2, 3, 5, 10, 15, 20, 30 years.

Worked example — PCA and forward buckets

Factor s.d. 12.96, 5.82, 2.14, 1.79: total variance = 209.62; Factor 1 share = $167.96/209.62 = 80.1\%$. Portfolio exposures +10, -20, -30 to the first three: $\sigma = \sqrt{129.6^2 + 116.4^2 + 64.2^2} = 185.65$.

3-yr 4.5% semiannual bond, flat 3.5%: value = **102.8245**. Forward-bucket '01s (0–1, 1–2, 2–3 yrs) = **0.0100, 0.0096, 0.0092**; all buckets together $\approx$ **0.0287**. Bucket durations = $0.0100/102.8245 \times 10,000 = 0.97$, then **0.93, 0.89**.

Key-rate hedge (illustrative): portfolio KR01 = 500 (2-yr) and 1,200 (10-yr) per bp; 2-yr and 10-yr par bonds have KR01 0.0190 and 0.0850 per 100 face (each only at its own key rate) $\Rightarrow$ short **2.63m** 2-yr and **1.41m** 10-yr.

14. Binomial Trees

One- and two-step trees, replication, risk-neutral valuation, delta, American options, dividends, currencies, futures

14.1 One-step model and risk-neutral valuation

$$u = e^{\sigma\sqrt{\Delta t}}, \quad d = \frac{1}{u} \quad \pi_u = \frac{e^{r\Delta t} - d}{u - d}, \quad \pi_d = 1 - \pi_u$$

$$c = e^{-r\Delta t} [\pi_u c_u + \pi_d c_d] \quad \Delta = \frac{c_u - c_d}{S_u - S_d} \quad \text{two-step: } c = e^{-2r\Delta t} [\pi_u^2 c_{uu} + 2\pi_u \pi_d c_{ud} + \pi_d^2 c_{dd}]$$

π_u, π_d are **risk-neutral** probabilities, not real ones. Replication: long Δ shares + short 1 call is riskless, so it earns r . $1/\Delta$ = calls to buy per share shorted. Higher $\sigma \Rightarrow$ wider tree. As steps $\rightarrow \infty$ the binomial price converges to Black–Scholes–Merton.

14.2 Other underlyings and American options

$$\text{Dividend yield } q: \pi_u = \frac{e^{(r-q)\Delta t} - d}{u - d} \quad \text{Currency: } \pi_u = \frac{e^{(r_{DC} - r_{FC})\Delta t} - d}{u - d} \quad \text{Futures: } \pi_u = \frac{1 - d}{u - d}$$

Index options: treat like a stock with dividend yield q . **American**: at every node (including today) value = max(exercise now, discounted expected value) — work backwards.

Worked example — Binomial valuations

Replication: $S = 100, S_u = 200, S_d = 50, X = 125, r = 8\%$: $\Delta = (75 - 0)/150 = 0.5$; riskless payoff = 25; $50 - c = 25e^{-0.08} = 23.08 \Rightarrow c = 26.92$.

Risk-neutral: $S = 20, \sigma = 14\%, r = 4\%, X = 20, 1 \text{ yr}$: $u = 1.1503, d = 0.8694, \pi_u = (1.0408 - 0.8694)/0.2809 = 0.610$; $c = 0.610 \times 3.005 \times e^{-0.04} = 1.76$; put = $1.76 + 20e^{-0.04} - 20 = 0.98$.

Two steps (2 yrs): $S_{uu} = 26.46, c = 0.610^2 \times 6.46 \times e^{-0.08} = 2.22$; put = $2.22 + 20e^{-0.08} - 20 = 0.68$.

American put: $S = 10, u = 1.2, d = 0.833, r = 2\%, \pi_u = 0.51, X = 12, 2 \text{ steps}$: terminal $14.40/10.00/6.94 \Rightarrow$ payoffs 0/2.00/5.06. Up node = $e^{-0.02}(0.49 \times 2) = 0.96$ (exercise = 0). Down node = $e^{-0.02}(0.51 \times 2 + 0.49 \times 5.06) = 3.43$ vs. exercise $12 - 8.33 = 3.67 \Rightarrow$ exercise

early. Today = $e^{-0.02}(0.51 \times 0.96 + 0.49 \times 3.67) = \mathbf{2.24}$ (> 2.00 , so do not exercise today).

15. The Black–Scholes–Merton Model

Lognormal prices, historical & implied volatility, BSM formulas, dividends, Black's model, early exercise, warrants

15.1 Lognormal property and volatility

$$\ln S_T \sim N \left[\ln S_0 + \left(\mu - \frac{\sigma^2}{2} \right) T, \sigma^2 T \right] \quad \boxed{E(S_T) = S_0 e^{\mu T}}$$

Continuously compounded annual return over T : mean = $\mu - \frac{\sigma^2}{2}$, s.d. = $\frac{\sigma}{\sqrt{T}}$ $\sigma_{\text{annual}} = \sigma_{\text{period}} \sqrt{\text{periods per year}}$

Prices are lognormal; continuously compounded returns are normal. Historical volatility: s.d. of $\ln(S_i/S_{i-1}) \times \sqrt{252}$ (use about as much history as the forecast horizon; drop ex-dividend days). Realised multi-period return = chain-linked (geometric).

15.2 BSM formulas

$$\boxed{c = S_0 N(d_1) - X e^{-rT} N(d_2)} \quad \boxed{p = X e^{-rT} N(-d_2) - S_0 N(-d_1)}$$

$$d_1 = \frac{\ln(S_0/X) + (r + \sigma^2/2)T}{\sigma\sqrt{T}} \quad d_2 = d_1 - \sigma\sqrt{T} \quad \text{put-call parity: } c + X e^{-rT} = p + S_0$$

Assumptions: lognormal prices; constant known r and σ ; continuous trading; no frictions, taxes or short-sale limits; no dividends; European exercise. $N(-x) = 1 - N(x)$.

15.3 Income, currencies, futures, early exercise, warrants

Dividend yield: replace S_0 by $S_0 e^{-qT}$ Discrete dividends: $S_0 - PV(D)$ Currency: $q = r_f$

Black's model (futures): $c = e^{-rT} [F_0 N(d_1) - X N(d_2)]$, $d_1 = \frac{\ln(F_0/X) + \sigma^2 T/2}{\sigma\sqrt{T}}$

American call: exercise just before ex-dividend only if $D > X [1 - e^{-r(T-t_n)}]$

$$\boxed{\text{Warrant} = \frac{N}{N+M} \times c}$$

Dividends lower call values and raise put values. With no dividends an American call is never exercised early. Larger dividends make early exercise of American *puts* less likely. N = shares outstanding, M = warrants; cost of issue = $M \times$ warrant value; share price falls by $\frac{M}{N+M}c$ if no benefit is perceived. **Implied volatility**: the σ that makes BSM equal the market price (found by iteration); plotting it against strike gives the **volatility smile**, and against strike and maturity the **volatility surface**.

Worked example — BSM valuations

Lognormal: $S_0 = 25$, $\mu = 12\%$, $\sigma = 20\%$, 3 months: $\ln S_T \sim N(3.244, 0.1^2)$; 95% range $e^{3.244 \pm 1.96(0.1)} = \mathbf{21.07 \text{ to } 31.18}$. $E(S_T)$, $S_0 = 25$, $\mu = 20\%$, 6 months: $25e^{0.10} = \mathbf{27.63}$. Returns 5%, -4%, 9%, 6% $\Rightarrow$ realised = $1.05 \times 0.96 \times 1.09 \times 1.06 - 1 = \mathbf{16.46\%}$.

Vola call: $S = 50$, $X = 45$, $r = 5\%$, $T = 0.25$, $\sigma = 12\%$: $d_1 = [0.10536 + (0.05 + 0.0072)0.25]/0.06 = 1.99$, $d_2 = 1.93$; $N(d_1) = 0.9767$, $N(d_2) = 0.9732$: $c = 48.835 - 44.441 \times 0.9732 = \mathbf{5.59}$; put = $5.59 + 44.441 - 50 = \mathbf{0.03}$.

With $q = 2\%$: $S_0 e^{-qT} = 49.75 \Rightarrow c = \mathbf{5.34}$. Dividends of 1 at 2 and 5 months, $S = 100$, $X = 100$, $r = 7\%$, $\sigma = 20\%$, 6 months: $S^* = 100 - 0.9884 - 0.9713 = 98.04 \Rightarrow$ call falls from 7.43 to $\mathbf{6.26}$; put rises from 3.99 to $\mathbf{4.78}$.

Warrants: $N = 1\text{m}$, $M = 0.5\text{m}$, $c = 6$: warrant = $6 \times 1/1.5 = \mathbf{4}$; cost 2m $\Rightarrow$ share price 50 $\rightarrow \mathbf{48}$.

16. Option Sensitivities: the Greeks

Naked & covered positions, stop-loss, delta hedging, gamma, theta, vega, rho, portfolio Greeks, portfolio insurance

16.1 Greek formulas (European, non-dividend stock)

Greek	Call	Put
Delta $\Delta = \partial V / \partial S$	$N(d_1)$	$N(d_1) - 1$
Gamma $\Gamma = \partial^2 V / \partial S^2$	$\frac{N'(d_1)}{S_0 \sigma \sqrt{T}}$	same as call
Vega $= \partial V / \partial \sigma$	$S_0 \sqrt{T} N'(d_1)$	same as call
Theta $\Theta = \partial V / \partial t$	$-\frac{S_0 N'(d_1) \sigma}{2\sqrt{T}} - rX e^{-rT} N(d_2)$	$-\frac{S_0 N'(d_1) \sigma}{2\sqrt{T}} + rX e^{-rT} N(-d_2)$
Rho $= \partial V / \partial r$	$X T e^{-rT} N(d_2)$	$-X T e^{-rT} N(-d_2)$

$$N'(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \quad \text{Dividend yield: } \Delta_{\text{call}} = e^{-qT} N(d_1), \quad \Delta_{\text{put}} = e^{-qT} [N(d_1) - 1]$$

$$\Delta_{\text{forward}} = 1 \quad (e^{-qT} \text{ with yield } q) \quad \Delta_{\text{futures}} = e^{rT} \quad (e^{(r-q)T} \text{ with yield } q)$$

Theta is per year: divide by 365 (calendar day) or 252 (trading day). Vega and rho are per 1.00 (100%) change: multiply by 0.01 for a 1% change.

16.2 Hedging relationships

$$\text{Calls to sell per long share} = \frac{1}{\Delta_c} \quad \text{Puts to buy per long share} = \frac{1}{|\Delta_c - 1|} \quad \boxed{\text{Options for gamma-neutral} = -\frac{\Gamma_P}{\Gamma_T}}$$

$$\Delta \Pi \approx \Delta \delta S + \frac{1}{2} \Gamma (\delta S)^2 + \Theta \delta t \quad \boxed{\Theta + rS\Delta + \frac{1}{2} \sigma^2 S^2 \Gamma = r\Pi} \quad \text{delta-neutral: } \Theta + \frac{1}{2} \sigma^2 S^2 \Gamma = r\Pi$$

Portfolio delta, gamma and vega = position-weighted sums. Gamma-neutral needs **options** (stock and forwards have zero gamma); after adding them, re-hedge delta with the underlying. For a delta-neutral book, large positive theta goes with large negative gamma (theta is a proxy for gamma).

Worked example — Greeks and hedges

Vola call ($S = 50, X = 45, r = 5\%, T = 0.25, \sigma = 12\%, d_1 = 1.99, N'(1.99) = 0.05508$): $\Delta = \mathbf{0.9767}$; $\Gamma = 0.05508 / (50 \times 0.12 \times 0.5) = \mathbf{0.0184}$; vega $= 50 \times 0.5 \times 0.05508 = \mathbf{1.38}$ (+0.0138 per 1% vol); $\Theta = -2.49/\text{yr} \Rightarrow -0.00988$ per trading day; rho $= 45 \times 0.25 \times e^{-0.0125} \times 0.9732 = \mathbf{10.81}$ (+0.108 per 1% rate).

Delta hedge: own 60,000 shares, call $\Delta = 0.60$: sell $60,000/0.6 = \mathbf{100,000}$ calls; if $\Delta \rightarrow 0.62$: $60,000/0.62 = \mathbf{96,774}$ (buy back $\approx 3,226$). With puts ($\Delta_p = -0.40$): buy $60,000/0.40 = \mathbf{150,000}$ puts.

Gamma: book $\Gamma = -6,000$, traded call $\Gamma = 1.25, \Delta = 0.6$: buy $6,000/1.25 = \mathbf{4,800}$ calls $\Rightarrow$ delta $+2,880 \Rightarrow$ **sell 2,880 shares**.

Naked call: sell 10,000 calls, $X = 23$, premium 4 (income 40,000); stock ends at 30: payoff $-70,000 \Rightarrow$ net $-30,000$.

Must-know facts — Greeks and hedging

Naked vs. covered	Naked (short call, no stock): unlimited loss. Covered (short call + stock): big loss if the stock falls.
Stop-loss	Buy the stock when it rises above X , sell when it falls below. Fails because of transaction costs and price uncertainty around X .
Static vs. dynamic	Static (hedge-and-forget) works only for small moves; dynamic hedging rebalances as delta changes (costly).
Where Greeks peak	Gamma, vega and theta are largest at the money , especially near expiry (vega is largest for longer-dated ATM options). Theta is usually negative (an ITM European put can be positive). Rho matters more for ITM options; small for equity options.
Limits & practice	Traders keep delta near zero daily and monitor gamma and vega against limits. Option sellers are short gamma and vega. Other Greeks: charm, vanna, vomma.
Portfolio insurance	Underlying + long puts (floor with upside). Synthetic put: sell index futures in the proportion given by the put delta — cheaper and more liquid, but must be rebalanced.

17. One-Page Formula Recap

Quick revision the night before the exam

VaR / ES (normal)	$(z\sigma - \mu)V$; $\left(\sigma \frac{N'(z)}{1-X} - \mu\right)V$; $z_{95} = 1.645, z_{99} = 2.326$; $\text{VaR}_T = \text{VaR}_1 \sqrt{T}$
Delta-normal gamma	$\text{VaR}_P = \delta \text{VaR}_f$; $\Delta P \approx \delta \Delta S + \frac{1}{2} \gamma (\Delta S)^2$
Historical sim.	99% of 300 = 3rd worst; ES = mean of losses beyond VaR
EWMA / GARCH	$\lambda \sigma_{n-1}^2 + (1-\lambda)u_{n-1}^2$; $\omega + \alpha u_{n-1}^2 + \beta \sigma_{n-1}^2$; $V_L = \frac{\omega}{1-\alpha-\beta}$
Hazard rate	$\text{PD}(t) = 1 - e^{-\lambda t}$; conditional $\text{PD} = \frac{\text{uncond.}}{\text{survival}}$; $\text{EL} = \text{PD} \times \text{LGD}$
Credit portfolio	$\sigma_i = \sqrt{PD - PD^2} L(1 - RR)$; $\frac{\sigma_p}{nL} = \frac{\sigma \sqrt{1+(n-1)\rho}}{L\sqrt{n}}$
Vasicek	$\text{WCDR} = N\left[\frac{N^{-1}(PD) + \sqrt{\rho} N^{-1}(0.999)}{\sqrt{1-\rho}}\right]$; capital = $\text{EAD} \times \text{LGD} \times (\text{WCDR} - PD)$
Op risk	BIA 15% $\overline{GI}$; $\text{LC} = 7X + 7Y + 5Z$; $K = \text{BIC} \times \text{ILM}$; scale $(\text{Rev}_B / \text{Rev}_A)^{0.23}$
Pricing	T-bill = $100 - \frac{n}{360}Q$; dirty = clean + AI; $\text{AI} = \frac{c}{2} \times \frac{\text{days since}}{\text{days in period}}$
Rates	$R_c = m \ln(1 + R_m/m)$; $F = \frac{R_2 T_2 - R_1 T_1}{T_2 - T_1}$; par = $\frac{2(1-d_T)}{A_T}$
Returns / yields	$\frac{BV_t + C - BV_{t-1}}{BV_{t-1}}$; annuity $\frac{C}{y}[1 - (1+y)^{-T}]$; perpetuity $\frac{C}{y}$
DV01 / duration	$-\frac{\Delta P}{10,000 \Delta y}$; $HR = \frac{DV01_P}{DV01_H}$; $D = \frac{P_- - P_+}{2P_0 \Delta y}$; $C = \frac{P_+ + P_- - 2P_0}{P_0 \Delta y^2}$
Price change	$\frac{\Delta P}{P} \approx -D \Delta y + \frac{1}{2} C \Delta y^2$; duration hedge $P_H = -\frac{V D_V}{D_H}$
Key rates / PCA	$\sum KR01 = DV01$; $D_k = \frac{KR01_k}{V} 10,000$; $\sigma = \sqrt{\sum (f_i \sigma_i)^2}$
Binomial	$u = e^{\sigma \sqrt{\Delta t}}$; $\pi_u = \frac{e^{r\Delta t} - d}{u - d}$; $\Delta = \frac{c_u - c_d}{S_u - S_d}$; $c = e^{-r\Delta t} [\pi_u c_u + \pi_d c_d]$
BSM	$c = S_0 N(d_1) - X e^{-rT} N(d_2)$; $d_{1,2} = \frac{\ln(S/X) + (r \pm \sigma^2/2)T}{\sigma \sqrt{T}}$; $S_0 \rightarrow S_0 e^{-qT}$
Lognormal	$E(S_T) = S_0 e^{\mu T}$; return mean $\mu - \frac{\sigma^2}{2}$, s.d. $\frac{\sigma}{\sqrt{T}}$; warrant = $\frac{N}{N+M} c$
Greeks	$\Delta_c = N(d_1)$; $\Gamma = \frac{N'(d_1)}{S \sigma \sqrt{T}}$; vega = $S \sqrt{T} N'(d_1)$; rho_c = $X T e^{-rT} N(d_2)$; gamma-neutral $-\frac{\Gamma_P}{\Gamma_T}$

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